Problem 1
Calculate the gravitational potential energy Ω of a star in terms of its mass and radius (i.e., calculate the q factor in the expression $-q\dfrac{GM^2}{R}$), in the cases where the density profile of the star goes like $\rho \propto r^{-1}$ and $\rho \propto r^{-2}$ (you can write the density as $\rho = \rho_0 (r/r_0)^{-n}$ where ρ_0 and r_0 are constants). What is the star's total internal energy for these two cases assuming that it is static and made up of gas that follows the ideal gas law (again, in units of GM^2/R)?

Problem 2
Assume a gas mixture with hydrogen, helium, and metal mass fractions of X=0.7, Y=0.28, and Z=0.02. You can assume that, for metals, the ratio of nuclear mass number to nuclear charge number is 2.
a. Evaluate the mean molecular weight of the gas for a series of cases with different degrees of ionization, assuming the ideal gas law:
- neutral gas
- hydrogen ionized, helium singly ionized, metals neutral.
- hydrogen and helium fully ionized, metals neutral.
- fully ionized gas.

b. Nuclear reactions will convert hydrogen to helium, and in later phases helium can be converted into carbon and oxygen. Assuming full ionization, compute the mean molecular weight when:
- hydrogen is fully converted into helium (X=0, Y=0.98, Z=0.02)
- helium is converted into metals (Z=1)

Problem 3
Show that the equation of state for an ideal gas is $P = nkT$ even if the particles in the gas are highly relativistic.

Problem 4
a. Prepare a $\log\rho - \log T$ plot with the range: $-10 \le \log\rho \le 10$ (x-axis) and $4 \le \log T \le 10$ (y-axis). Write down the equations $\log T = f(\log\rho)$ that characterize the boundaries in this plane between different regimes in pressure. Specifically, where
- the radiation pressure is equal to the ideal gas pressure
- the ideal gas pressure is equal to the pressure of degenerate, non-relativistic electron gas
- the pressure of degenerate, non-relativistic electron gas is equal to the pressure of degenerate, relativistic electron gas.

Adopt: X=0, Y=1, Z=0 and assume that the gas is fully ionized.
Plot these three curves on your $\log\rho - \log T$ plot and label the regions on the plot where each type of pressure dominates.

b. The total gas pressure can be written as $P_{tot} = P_{rad} + P_{ion} + P_e$, where P_{rad} is the radiation pressure, P_{ion} is the ideal gas ion pressure, and P_e is the total electron pressure. You can approximate the electron pressure as the sum in quadrature of the non-degenerate and degenerate components: $P_e = \left(P_{e,nd}^2 + P_{e,d}^2\right)^{1/2}$, and assume that the degenerate electrons are non-relativistic. Write a computer routine that computes the total gas pressure for a given gas density ρ, temperature T, and mean molecular weights for ions and electrons μ_i and μ_e. Use your routine to make a plot of total gas pressure $\log P_{tot}$ versus density $\log \rho$ for the following values of temperature: $\log T = 5,6,7,8$, assuming the mean molecular weight from part a.

Problem 5

Important quantities of neutral (He I), singly ionized (He II), and doubly ionized (He III) helium can be present at the same time. For an ideal non-degenerate pure helium gas, write down the set of equations to solve for the fractions of He I, He II, and He III. You can treat the ionization potentials (χ_{II}, χ_{III}) and partition functions (G_I, G_{II}, G_{III}) as known arbitrary functions.