Problem 1

The energy-averaged cross section is given by $\langle \sigma v \rangle = C S_0 \tau^2 e^{-\tau}$, where C and S_0 are constants, τ is given by $\tau = \dfrac{19.721 W^{1/3}}{\left(T/10^7 K\right)^{1/3}}$ and $W = Z_j^2 Z_k^2 \dfrac{A_j A_k}{A_j + A_k}$, where Z, A are the charge and mass numbers of species j and k. The reaction rate (number of reactions per unit time, per unit volume) is given by $r_{j,k} = X_j X_k r'_{j,k}$, where X_j, X_k are the mass fractions of species j and k, and $r'_{j,k} = \dfrac{\rho^2}{\left(1+\delta_{jk}\right) A_j A_k m_p^2} \langle \sigma v \rangle$.

Consider the reactions:

$C_6^{12} + H_1^1 \rightarrow C_6^{13}$

$C_6^{13} + H_1^1 \rightarrow N_7^{14}$

$N_7^{14} + H_1^1 \rightarrow N_7^{15}$

$N_7^{15} + H_1^1 \rightarrow C_6^{12} + He_2^4$

with corresponding reaction rates $r_{1,12}, r_{1,13}, r_{1,14}, r_{1,15}$

a. Derive expressions for the above 4 reaction rates as functions of C, S_0, ρ, T, and the mass fractions X_i.

b. Evaluate the temperature exponent $\dfrac{\partial\left(\ln\langle \sigma v \rangle\right)}{\partial \ln T}$ for the above 4 equations at $T = 3 \times 10^7 K$.

Problem 2

a. Write down a set of coupled differential equations for d/dt of the mass fractions $C^{12}, C^{13}, N^{14}, N^{15}, H^1, He^4$ (I am using these symbols to signify mass fractions, instead of the standard X_j symbol), assuming that the reactions in problem 4 are the only ones occurring (e.g., N^{14} is only produced by the 2nd reaction and destroyed by the 3rd one).

b. Derive expressions for the relative abundances of C^{12}, C^{13}, N^{15} (in equilibrium) as functions of the N^{14} abundance.

c. Given that the total number of $C+N$ nuclei are conserved, derive an expression for the absolute equilibrium abundances $C^{12}, C^{13}, N^{14}, N^{15}$ as functions of the reaction rates and the total number of $C+N$ nuclei.

(*NOTE: for this problem, you can take the reaction rates as known functions of temperature, i.e., you don't need to evaluate the expressions for* $r'_{1,12}$, etc)