Problem 1
The energy-averaged cross section is given by $\langle \sigma v \rangle = C S_0 \tau^2 e^{-\tau}$, where $C$ and $S_0$ are constants, $\tau$ is given by $\tau = 19.721 W^{\frac{1}{3}}$ and $W = Z_j^2 Z_k^2 A_j A_k / (T/10^7 K)^{\frac{1}{3}}$, where $Z, A$ are the charge and mass numbers of species $j$ and $k$. The reaction rate (number of reactions per unit time, per unit volume) is given by $r_{j,k} = X_j X_k r'_{j,k}$, where $X_j, X_k$ are the mass fractions of species $j$ and $k$, and $r'_{j,k} = \rho^2 (1 + \delta_{jk}) A_j A_k m_p / \langle \sigma v \rangle$.

Consider the reactions:
\begin{align*}
C_6^{12} + H_1^1 & \rightarrow C_6^{13} \\
C_6^{13} + H_1^1 & \rightarrow N_7^{14} \\
N_7^{14} + H_1^1 & \rightarrow N_7^{15} \\
N_7^{15} + H_1^1 & \rightarrow C_6^{12} + He_2^4
\end{align*}

a. Derive expressions for the above 4 reaction rates as functions of $C, S_0, \rho, T,$ and the mass fractions $X_i$.

b. Evaluate the temperature exponent $\frac{\partial \langle \sigma v \rangle}{\partial \ln T}$ for the above 4 equations at $T = 3 \times 10^7 K$.

Problem 2
a. Write down a set of coupled differential equations for $d/dt$ of the mass fractions $C_6^{12}, C_6^{13}, N_7^{14}, N_7^{15}, H_1^1, He_2^4$ (I am using these symbols to signify mass fractions, instead of the standard $X_j$ symbol), assuming that the reactions in problem 4 are the only ones occurring (e.g., $N_7^{14}$ is only produced by the 2nd reaction and destroyed by the 3rd one).

b. Derive expressions for the relative abundances of $C_6^{12}, C_6^{13}, N_7^{14}$ (in equilibrium) as functions of the $N_7^{14}$ abundance.

c. Given that the total number of $C + N$ nuclei are conserved, derive an expression for the absolute equilibrium abundances $C_6^{12}, C_6^{13}, N_7^{14}, N_7^{15}$ as functions of the reaction rates and the total number of $C + N$ nuclei.

(NOTE: for this problem, you can take the reaction rates as known functions of temperature, i.e., you don’t need to evaluate the expressions for $r'_{j,k}$, etc)