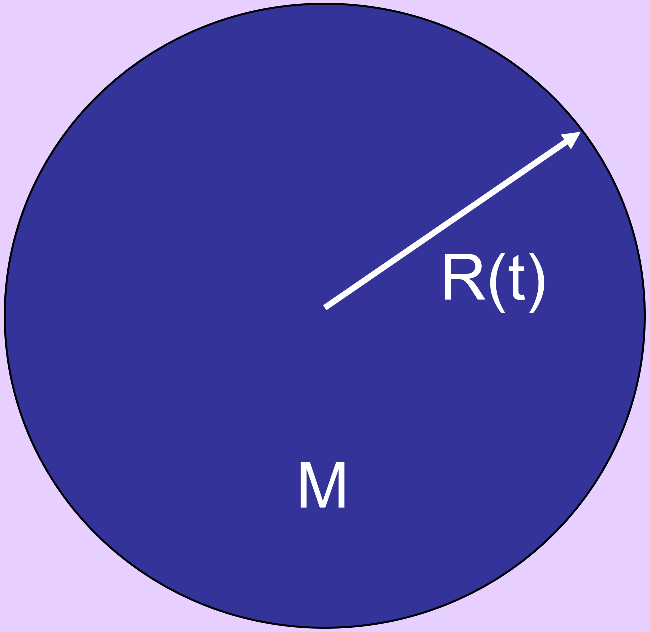
The Friedmann Equation

$$\ddot{R} = -\frac{GM}{R^2}$$

$$\ddot{R}\dot{R} = -\frac{GM\dot{R}}{R^2}$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{R}^2\right) = \frac{d}{dt}\left(\frac{GM}{R}\right)$$

$$\frac{1}{2}\dot{R}^2 = \frac{GM}{R} + K$$



Kinetic + potential energy per unit mass = constant

The Friedmann Equation

$$\frac{1}{2}\dot{R}^2 = \frac{GM}{R} + K$$

$$\boxed{M = \rho\frac{4}{3}\pi R^3}$$

$$\frac{1}{2}\dot{R}^2 = \frac{4\pi G\rho R^2}{3} + K$$

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} + \frac{2K}{R^2}$$

$$\boxed{a(t) = \frac{R(t)}{R(t=0)} = \frac{R}{R_0}}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{2K}{R_0^2 a^2}$$

$$\boxed{\dot{a} = \frac{\dot{R}}{R_0}}$$

The Friedmann Equation

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2K}{R_0^2}\frac{1}{a(t)^2}}$$

Newtonian form of Friedman equation

General Relativity:

- Replace density with energy density

$$\rho(t) \rightarrow \frac{\varepsilon(t)}{c^2}$$

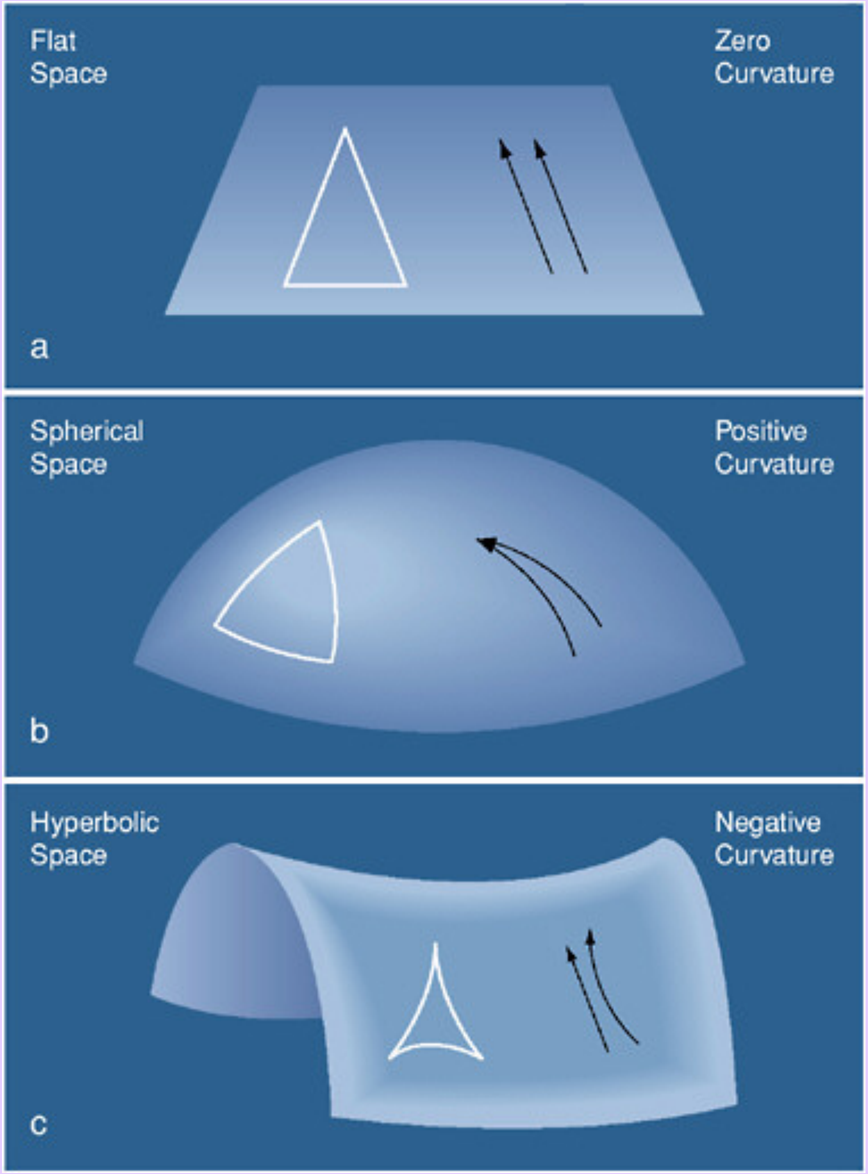
$$E = \left(m^2c^4 + p^2c^2\right)^{1/2}$$

- Constant of integration is curvature of spacetime

$$\frac{2K}{R_0^2} \rightarrow -\frac{kc^2}{R_0^2}$$

$$k = \begin{cases} -1 & \text{negative curvature} \\ 0 & \text{flat space} \\ +1 & \text{positive curvature} \end{cases}$$

The Friedmann Equation



k=0

k=+1

k=-1

The Friedmann Equation

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2}}$$

$$\frac{\dot{a}}{a} \equiv H(t)$$

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t)$$

If k=0

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G}H(t)^2$$

Critical density

$$\frac{\varepsilon_{c,0}}{c^2} = 2.8\times 10^{11} h^2 M_\odot \mathrm{Mpc}^{-3}$$

The Friedmann Equation

$$\boxed{\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}}$$

Energy density in units of critical density

$$H(t)^2 = H(t)^2\,\Omega(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2}$$

$$1 - \Omega(t) = -\frac{kc^2}{R_0^2}\frac{1}{H(t)^2\,a(t)^2}$$

$$\boxed{\Omega(t) \begin{cases} <1 & k=-1 \\ =1 & k=0 \\ >1 & k=+1 \end{cases}}$$

Sign of 1-Ω does not change as universe expands.

The Friedmann Equation

At the present epoch: $$H_0^2\left(1-\Omega_0\right) = -\frac{kc^2}{R_0^2}$$

Replace curvature constant in Friedman equation:

$$1-\Omega(t) = \frac{H_0^2\left(1-\Omega_0\right)}{H(t)^2\, a(t)^2}$$

$$\boxed{\frac{H(t)^2}{H_0^2}\left[1-\Omega(t)\right] = \frac{\left(1-\Omega_0\right)}{a(t)^2}}$$

The Fluid Equation

$$dQ = dE + PdV$$

1st law of thermodynamics

$$\dot{E} + P\dot{V} = 0$$

$$E(t) = \varepsilon(t)V(t)$$

$$\dot{\varepsilon}V + \varepsilon\dot{V} + P\dot{V} = 0$$

$$V(t) = \frac{4}{3}\pi R(t)^3 \rightarrow$$

$$\dot{V} = \frac{4}{3}\pi 3R^2\dot{R} \rightarrow$$

$$V\left(\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P)\right) = 0$$

$$\dot{V} = V3\frac{\dot{R}}{R} = V3\frac{\dot{a}}{a}$$

$$\boxed{\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0}$$

The Acceleration Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon(t)a(t)^2 - \frac{kc^2}{R_0^2}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\varepsilon}a^2 + 2\varepsilon a\dot{a}\right)$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\varepsilon}\frac{a}{\dot{a}} + 2\varepsilon\right)$$

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

$$\dot{\varepsilon}\frac{a}{\dot{a}} = -3(\varepsilon + P)$$

The Acceleration Equation

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(-3\left(\varepsilon + P\right) + 2\varepsilon\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\varepsilon + 3P\right)}$$

The energy density is always positive

- If Pressure is positive then the universe must decelerate.
  (e.g., baryonic gas, photons, dark matter)

- If Pressure is negative, the universe can accelerate.
  (e.g., dark energy)

The Equations of Motion of the Universe

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2}}$$

Friedmann Equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)}$$

Acceleration Equation

Two equations and three unknowns: $a(t),\ \varepsilon(t),\ P(t)$

Need a third equation: Equation of state $P = P(\varepsilon)$

$$\boxed{P = w\varepsilon}$$

The Equation of State

$$\boxed{P = w\varepsilon}$$

- Non-relativistic particles: matter (Ideal gas law)

$$P = \frac{\rho}{\mu}kT = \frac{kT}{\mu c^2}\varepsilon \qquad w = \frac{kT}{\mu c^2} \approx 0$$

- Relativistic particles: radiation

$$P = \frac{1}{3}\varepsilon \qquad w = \frac{1}{3}$$

- Mildly relativistic particles:

$$0 < w < \frac{1}{3}$$

Cosmological Constant / Dark Energy

1915 Einstein's GR equations predict a dynamic universe.

1917 But Einstein thought the Universe was static, so he introduced the "Cosmological Constant", **Λ**, to his equations of motion.

1929 When Hubble discovered the expansion of the universe, Einstein called **Λ** his "greatest blunder".

1998 SN results show that the universe is accelerating in its expansion so scientists revive **Λ**

The cosmological constant or "Dark Energy" is thought to be the energy of a vacuum, predicted by quantum mechanics.

Cosmological Constant / Dark Energy

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} + \frac{\Lambda}{3}}$$

Friedman Equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda}{3}}$$

Acceleration Equation

$$\boxed{\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0}$$

Fluid Equation:

If the energy density of Dark Energy is constant with time

$$\dot{\varepsilon} = -3\frac{\dot{a}}{a}(\varepsilon + P) = 0 \rightarrow P = -\varepsilon \rightarrow \quad \boxed{w = -1}$$

Cosmological Constant / Dark Energy

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} + \frac{\Lambda}{3}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) + \frac{8\pi G}{3c^2}\left(\frac{c^2\Lambda}{8\pi G}\right) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\varepsilon_m + \varepsilon_\Lambda\right) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2}$$

$$\boxed{H(t)^2\left(1 - \Omega_m - \Omega_\Lambda\right) = -\frac{kc^2}{R_0^2}\frac{1}{a(t)^2}}$$

Cosmological Parameters

Cosmological parameter #4: the matter density

$$\Omega_m$$

$$\boxed{\Omega_m \approx 0.28 \pm 0.01}$$

Cosmological Parameters

Cosmological parameter #5: the baryon density

$$\Omega_b$$

$$\boxed{\Omega_b \approx 0.047 \pm 0.001}$$

Cosmological Parameters

<u>Cosmological parameter #6</u>: the dark energy density

$$\Omega_\Lambda$$

$$\boxed{\Omega_\Lambda \approx 0.72 \pm 0.01}$$

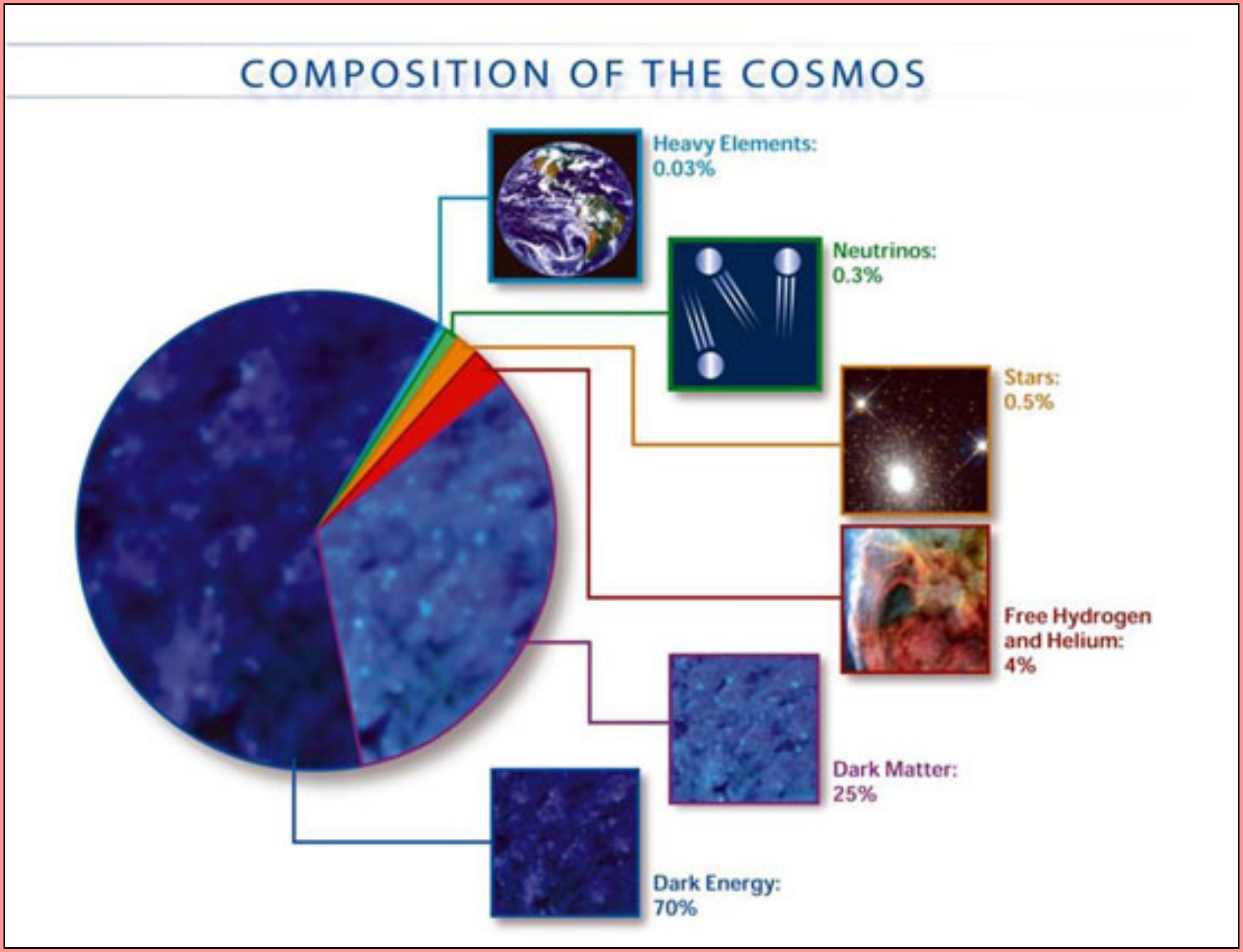
Cosmological Parameters

Cosmological parameter #7: the radiation density

$$\Omega_\gamma \qquad \Omega_\nu$$

$$\boxed{\Omega_\gamma \approx 5\times 10^{-5}} \qquad \boxed{\Omega_\nu \approx 3.4\times 10^{-5}}$$

Cosmological Parameters



The Evolution of Energy Density

$$\boxed{\dot{\varepsilon} + 3\frac{\dot{a}}{a}\left(\varepsilon + P\right) = 0}$$

$$\boxed{P = w\varepsilon}$$

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}\left(1 + w\right)\varepsilon = 0$$

$$\frac{d\varepsilon}{dt} = -\frac{da}{dt}\frac{3}{a}\left(1 + w\right)\varepsilon$$

$$\frac{d\varepsilon}{\varepsilon} = -\frac{da}{a}3\left(1 + w\right)$$

If w is constant with a:

$$\ln\varepsilon = \ln\varepsilon_0 - 3\left(1 + w\right)\ln a$$

$$\boxed{\varepsilon = \varepsilon_0 a^{-3\left(1+w\right)}}$$

The Evolution of Energy Density

$$\boxed{\varepsilon = \varepsilon_0 a^{-3(1+w)}}$$

- Non-relativistic particles (baryons, dark matter)

$$w = 0 \quad \rightarrow \quad \boxed{\varepsilon = \varepsilon_0 a^{-3} = \varepsilon_0 (1+z)^3}$$

- Relativistic particles (photons, neutrinos)

$$w = \frac{1}{3} \quad \rightarrow \quad \boxed{\varepsilon = \varepsilon_0 a^{-4} = \varepsilon_0 (1+z)^4}$$

- Dark energy

$$w = -1 \quad \rightarrow \quad \boxed{\varepsilon = \varepsilon_0}$$

The Friedmann Equation

$$\frac{H(t)^2}{H_0^2} = \frac{H(t)^2}{H_0^2}\Omega(t) + \frac{(1-\Omega_0)}{a(t)^2}$$

$$\left\{ \begin{array}{c} \varepsilon_c = \dfrac{3c^2}{8\pi G} H^2 \\ \dfrac{H^2}{H_0^2}\dfrac{\varepsilon}{\varepsilon_c} = \dfrac{\varepsilon_c}{\varepsilon_{c,0}}\dfrac{\varepsilon}{\varepsilon_c} = \dfrac{\varepsilon}{\varepsilon_{c,0}} \end{array} \right\}$$

$$\boxed{\frac{H^2}{H_0^2} = \frac{\varepsilon}{\varepsilon_{c,0}} + \frac{(1-\Omega_0)}{a^2}}$$

$$\boxed{\varepsilon = \varepsilon_m + \varepsilon_r + \varepsilon_\Lambda = \frac{\varepsilon_{m,0}}{a^3} + \frac{\varepsilon_{r,0}}{a^4} + \varepsilon_{\Lambda,0}}$$

The Friedmann Equation

$$\frac{H^2}{H_0^2} = \frac{1}{\varepsilon_{c,0}}\left(\frac{\varepsilon_{m,0}}{a^3} + \frac{\varepsilon_{r,0}}{a^4} + \varepsilon_{\Lambda,0}\right) + \frac{(1-\Omega_0)}{a^2}$$

$$\boxed{\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}}$$

$$\boxed{\Omega_0 = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0}}$$

$$\boxed{1-\Omega_0 = \Omega_{k,0}}$$

$$\boxed{\Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0} + \Omega_{k,0} = 1}$$

Cosmological parameter #8: the spatial curvature

$$\Omega_k$$

$$\boxed{\Omega_k = -0.002 \pm 0.004}$$

Cosmological Parameters

Cosmological parameter #9: the equation of state of dark energy

$$w$$

$$\boxed{w \approx -1.04 \pm 0.07}$$

The Friedmann Equation

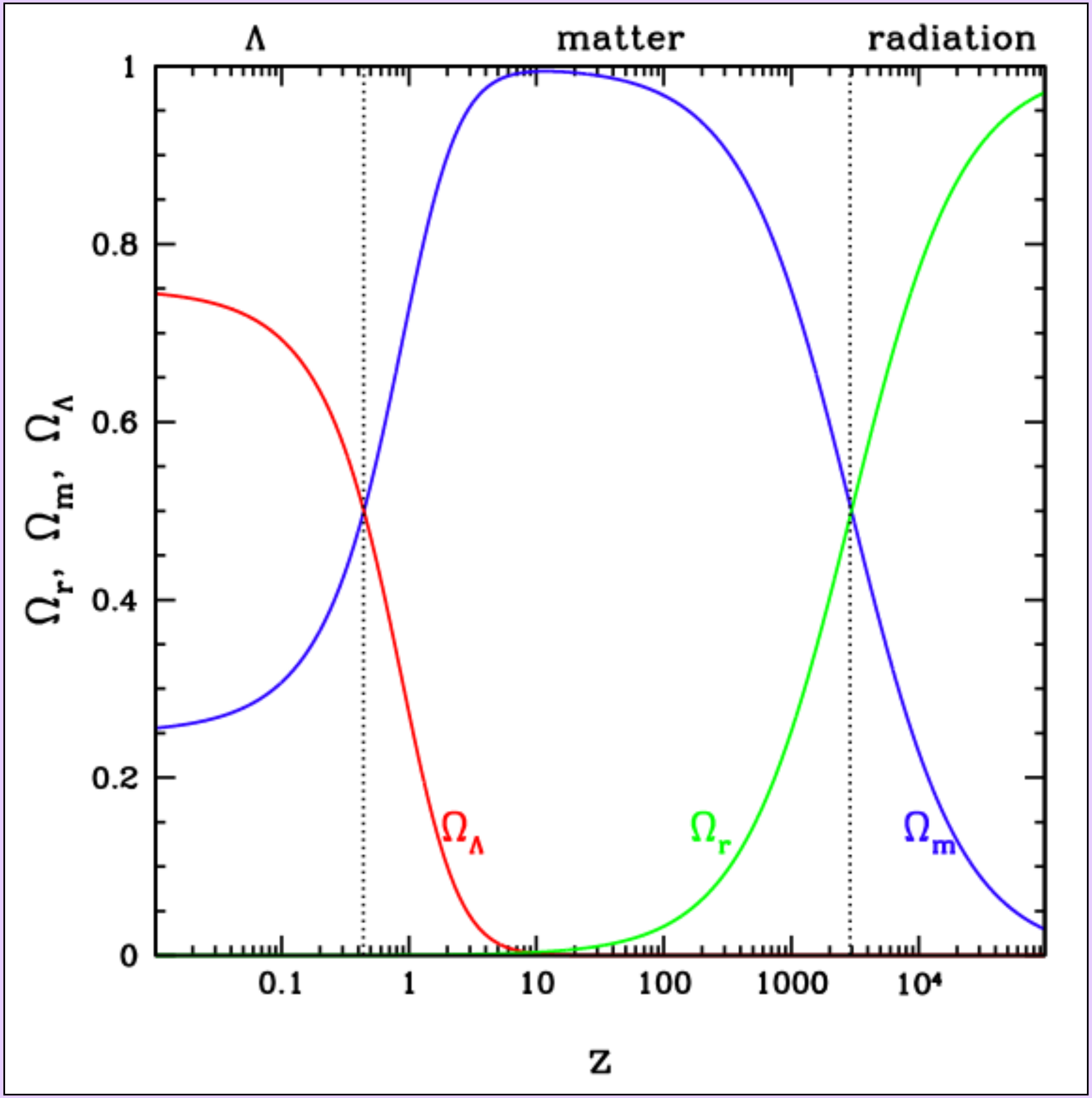
$$\boxed{\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{k,0}}{a^2} + \frac{\Omega_{\Lambda,0}}{a^{3(1+w)}}}$$

$$\Omega_m(z) = \frac{\varepsilon_m}{\varepsilon_c} = \frac{\varepsilon_{m,0}}{a^3\varepsilon_c} = \frac{\varepsilon_{m,0}}{a^3\varepsilon_{c,0}}\frac{\varepsilon_{c,0}}{\varepsilon_c} = \frac{\Omega_{m,0}}{a^3}\frac{{H_0}^2}{H^2}$$

$$\Omega_r(z) = \frac{\Omega_{r,0}}{a^4}\frac{{H_0}^2}{H^2}$$

$$\Omega_\Lambda(z) = \frac{\Omega_{\Lambda,0}}{a^{3(1+w)}}\frac{{H_0}^2}{H^2}$$

The Friedmann Equation



The Friedmann Equation

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{k,0}}{a^2} + \frac{\Omega_{\Lambda,0}}{a^{3(1+w)}}$$

Solving this equation gives expansion history a(t)
(which also specifies the age of the universe t_0)

Solving the Friedmann Equation

Special case #1: empty universe

$$\boxed{\Omega_{k,0} = 1}$$

$$\boxed{\frac{H^2}{H_0^2} = \cancelto{0}{\frac{\Omega_{r,0}}{a^4}} + \cancelto{0}{\frac{\Omega_{m,0}}{a^3}} + \frac{\Omega_{k,0}}{a^2} + \cancelto{0}{\frac{\Omega_{\Lambda,0}}{a^{3(1+w)}}}}$$

$$\frac{H^2}{H_0^2} = \frac{1}{a^2} \quad \rightarrow H = H_0 \frac{1}{a} \quad \rightarrow \dot{a} = H_0 \quad \rightarrow \int_0^a da' = \int_0^t H_0 dt'$$

$$\boxed{a(t) = H_0 t}$$

$$\boxed{t_0 = \frac{1}{H_0}}$$

Solving the Friedmann Equation

<u>Special case #2: radiation dominated universe</u>

$$\boxed{\Omega_{r,0} = 1}$$

$$\boxed{\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \cancelto{0}{\frac{\Omega_{m,0}}{a^3}} + \cancelto{0}{\frac{\Omega_{k,0}}{a^2}} + \cancelto{0}{\frac{\Omega_{\Lambda,0}}{a^{3(1+w)}}}}$$

$$\frac{H^2}{H_0^2} = \frac{1}{a^4} \rightarrow H = H_0 \frac{1}{a^2} \rightarrow \dot{a} = H_0 \frac{1}{a} \rightarrow \int_0^a a' da' = \int_0^t H_0 dt'$$

$$\boxed{a(t) = \left(2H_0 t\right)^{1/2}}$$

$$\boxed{t_0 = \frac{1}{2H_0}}$$

Solving the Friedmann Equation

Special case #3: matter dominated universe

$$\Omega_{m,0} = 1$$

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{k,0}}{a^2} + \frac{\Omega_{\Lambda,0}}{a^{3(1+w)}}$$

0 0 0

$$\frac{H^2}{H_0^2} = \frac{1}{a^3} \rightarrow H = H_0 \frac{1}{a^{3/2}} \rightarrow \dot{a} = H_0 \frac{1}{\sqrt{a}} \rightarrow \int_0^a \sqrt{a'}\,da' = \int_0^t H_0\,dt'$$

$$a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3}$$

$$t_0 = \frac{2}{3H_0}$$

Solving the Friedmann Equation

Special case #4: dark energy dominated universe $\boxed{\Omega_{\Lambda,0} = 1}$

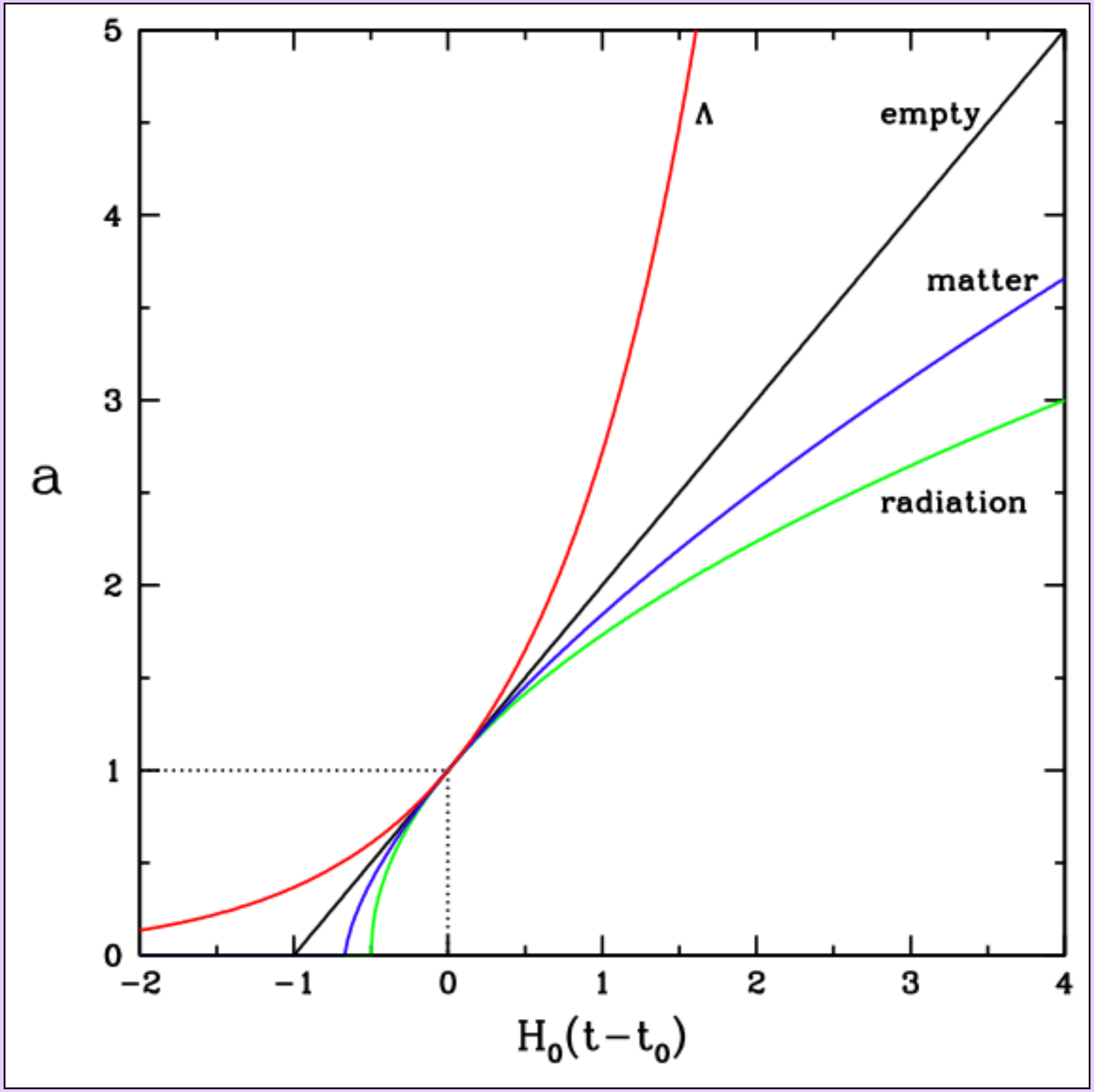
$$\frac{H^2}{H_0^2} = \cancelto{0}{\frac{\Omega_{r,0}}{a^4}} + \cancelto{0}{\frac{\Omega_{m,0}}{a^3}} + \cancelto{0}{\frac{\Omega_{k,0}}{a^2}} + \Omega_{\Lambda,0}$$

$$\frac{H^2}{H_0^2} = 1 \quad \rightarrow H = H_0 \quad \rightarrow \dot{a} = H_0 a \quad \rightarrow \int_a^1 \frac{da'}{a'} = \int_t^{t_0} H_0 dt'$$

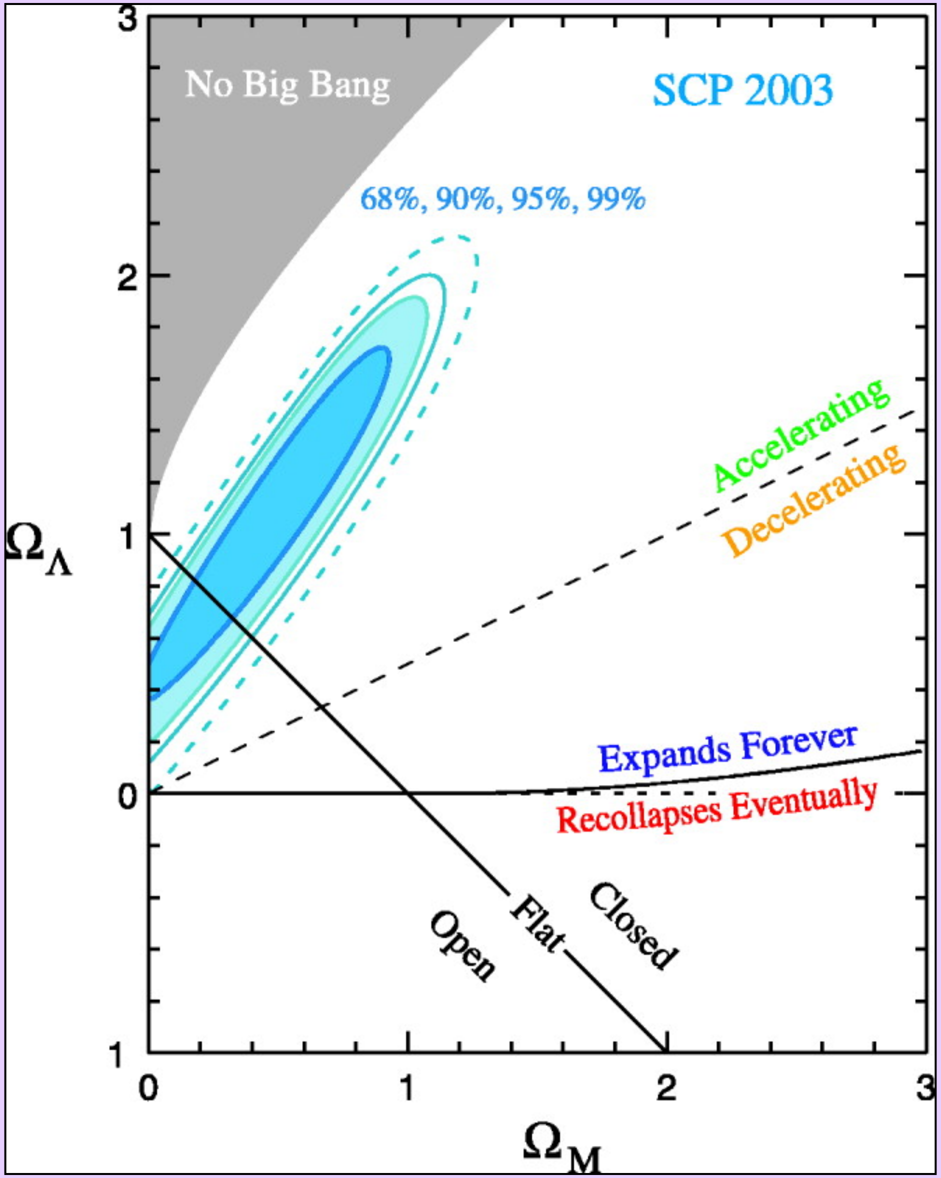
$$\boxed{a(t) = e^{H_0 (t - t_0)}}$$

$$\boxed{t_0 = \infty}$$

Solving the Friedmann Equation



Solving the Friedmann Equation



Knop et al. (2003)

Distance Measures in Cosmology

Hubble time: Time it took the universe to reach its present size if expansion rate has always been the same.

$$t_H \equiv \frac{1}{H_0} = 9.78 h^{-1} \mathrm{Gyr}$$

Hubble distance: Distance light travels in a Hubble time.

$$D_H \equiv \frac{c}{H_0} = 3000 h^{-1} \mathrm{Mpc}$$

Proper distance: Distance measured in rulers between two points.

$$D_P$$

Distance Measures in Cosmology

- Proper distance between point at *a* and *a+da:*

$$dD_P = c \times \frac{da}{\dot{a}}$$

$$= c\frac{da}{a\left(\dfrac{\dot{a}}{a}\right)} = c\frac{da}{aH} = \frac{c}{H_0}\frac{da}{a}\left(\frac{H}{H_0}\right)^{-1}$$

$$= D_H \frac{dz}{(1+z)E(z)}$$

- Integrating: $\boxed{D_P = D_H \int_0^z \frac{dz'}{(1+z')E(z')}}$

The Friedmann Equation

$$\frac{H^2}{H_0^2} = \cancelto{0}{\frac{\Omega_{r,0}}{a^4}} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{k,0}}{a^2} + \frac{\Omega_{\Lambda,0}}{a^{3(1+w)}}$$

$$\frac{H}{H_0} = \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{3(1+w)}} = E(z)$$

Distance Measures in Cosmology

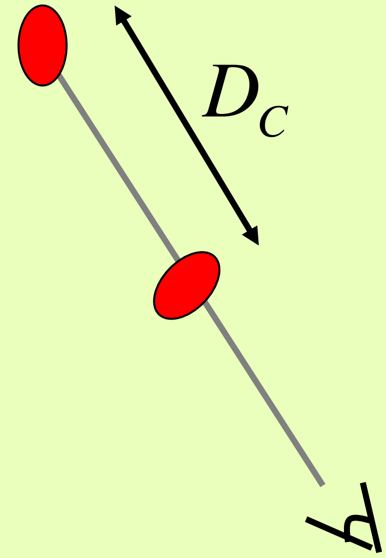
Comoving distance: Distance between points that remains constant if both points move with the hubble flow.

$$D_C \equiv \frac{D_P}{a} = D_P\left(1+z\right)$$

$$\boxed{D_C = D_H \int_0^z \frac{dz'}{E\left(z'\right)}}$$

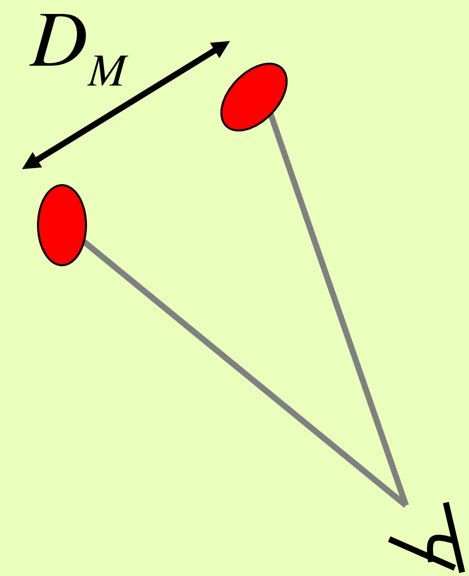
Distance Measures in Cosmology

<u>Comoving distance</u> (line-of-sight)



$$D_C = D_H \int_0^z \frac{dz'}{E(z')}$$

<u>Comoving distance</u> (transverse)

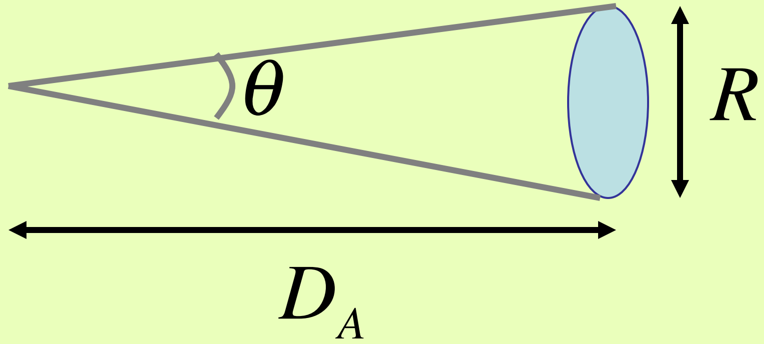


$$D_M = \begin{cases} D_H \dfrac{1}{\sqrt{\Omega_k}} \sinh\left(\sqrt{\Omega_k}\, D_C / D_H\right) & \text{for } \Omega_k > 0 \\ D_C & \text{for } \Omega_k = 0 \\ D_H \dfrac{1}{\sqrt{|\Omega_k|}} \sin\left(\sqrt{\Omega_k}\, D_C / D_H\right) & \text{for } \Omega_k < 0 \end{cases}$$

Distance Measures in Cosmology

Angular diameter distance: Ratio of object's physical size to angular size.

$$\boxed{D_A = \frac{D_M}{(1+z)}}$$



$$D_A = \frac{R}{\theta}$$

Distance Measures in Cosmology

<u>Luminosity distance</u>: Distance that defines the relationship between luminosity and flux.

$$D_L = \sqrt{\frac{L}{4\pi F}}$$

$$\boxed{D_L = (1+z)\,D_M = (1+z)^2\,D_A}$$

Distance Measures in Cosmology

<u>Comoving volume</u>: Volume in which densities of non-evolving objects are constant with redshift.

$$dV_C = D_H \frac{(1+z)^2 D_A^2}{E(z)} d\Omega dz$$

$$\boxed{V_C = \frac{4\pi}{3} D_M^3 \quad \text{for } \Omega_k = 0}$$

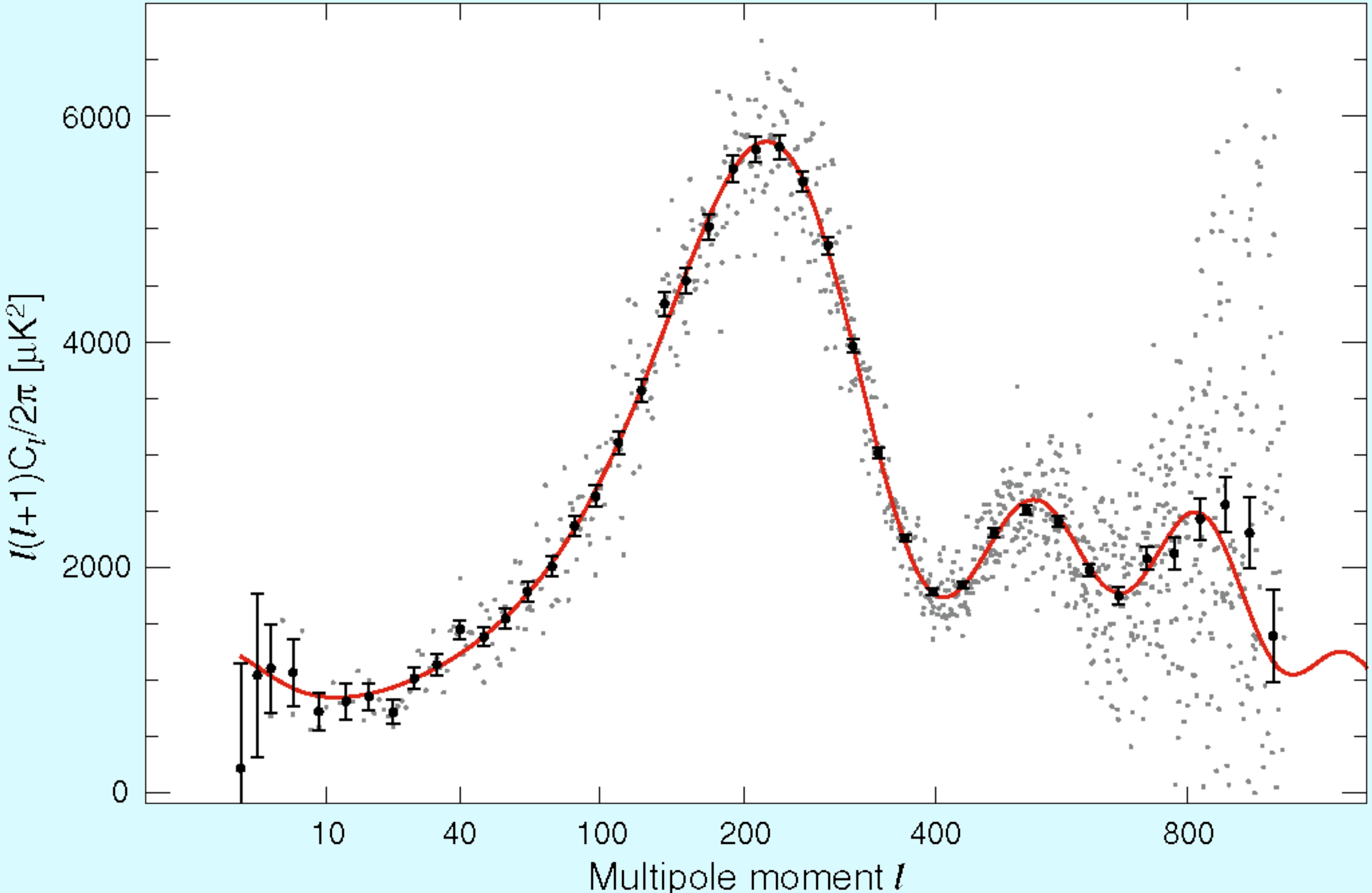
Distance Measures in Cosmology

<u>Lookback time</u>: Difference between age of universe now and age of universe at the time photons were emitted from object.

$$\boxed{t_L = t_H \int_0^z \frac{dz'}{(1+z')E(z')}}$$

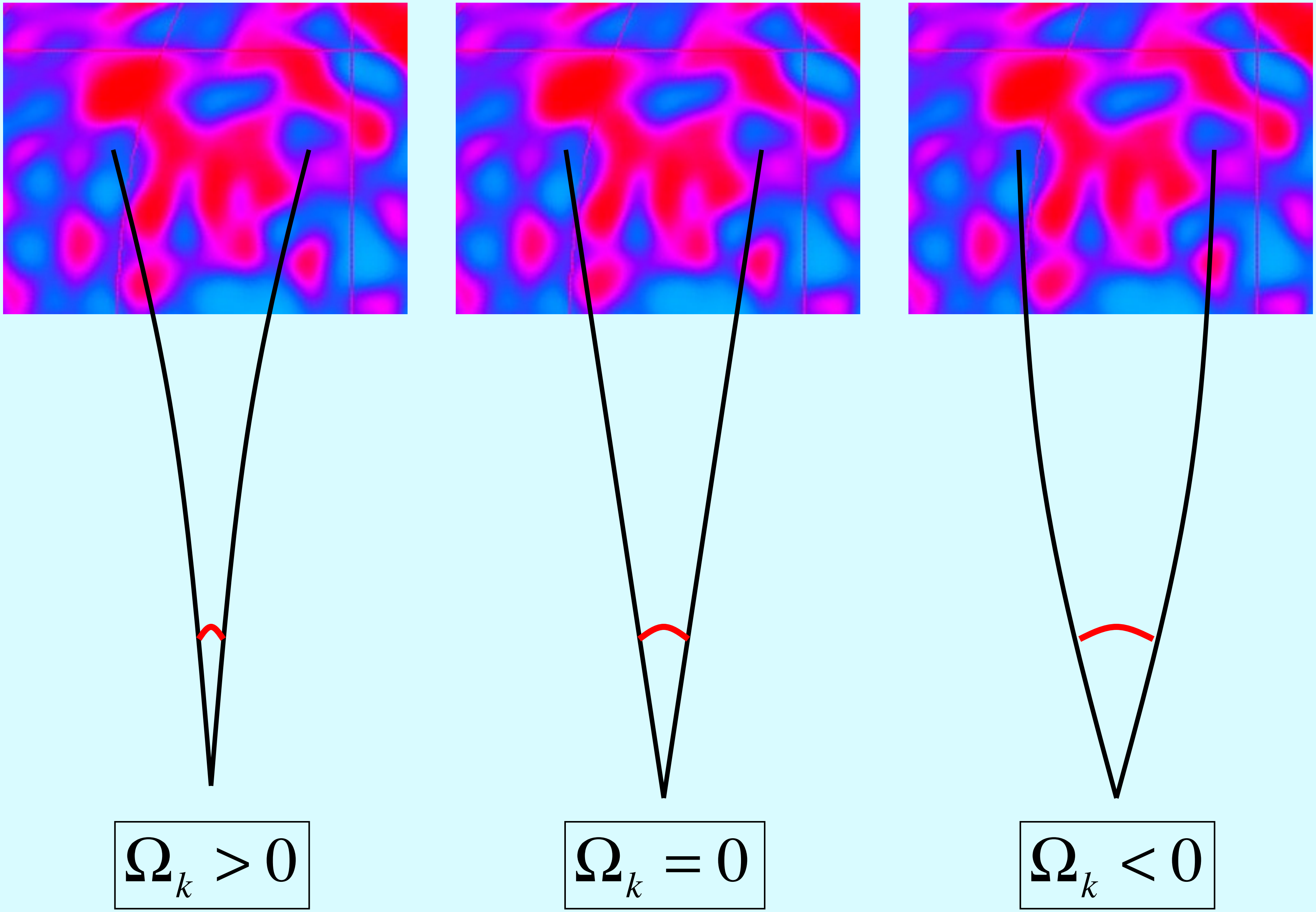
Age of the universe at redshift z: $t_0 - t_L(z)$

Constraining Cosmological Parameters



Dunkley et al. (2009)

Constraining Cosmological Parameters



Constraining Cosmological Parameters

Measurement

Physics

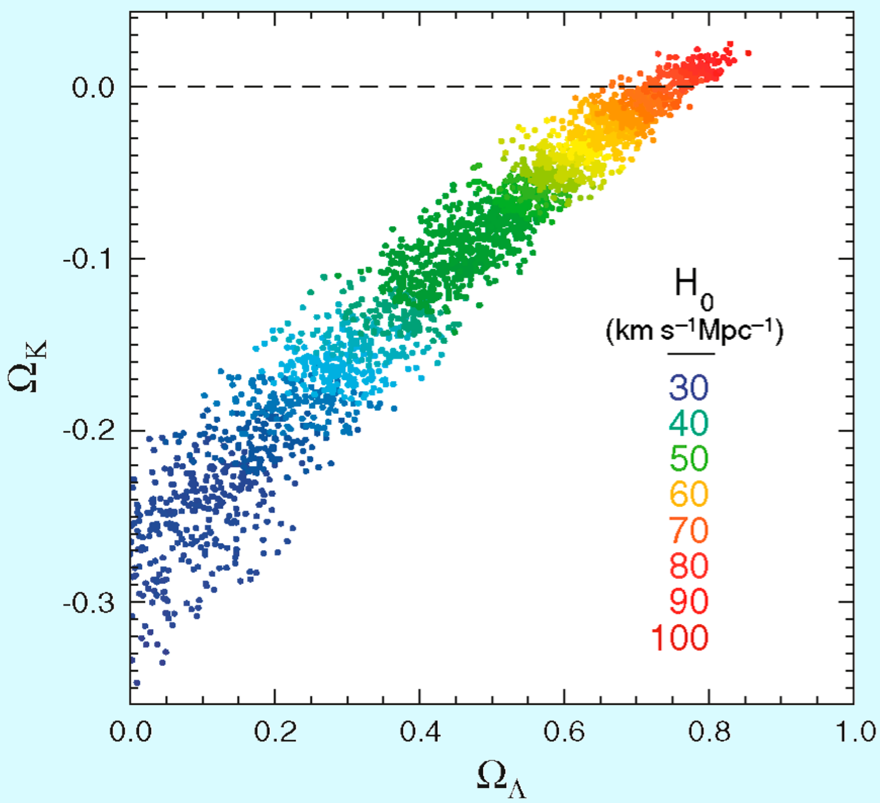
$$D_A = \frac{\lambda}{\theta} = \frac{l}{2\pi}\lambda$$

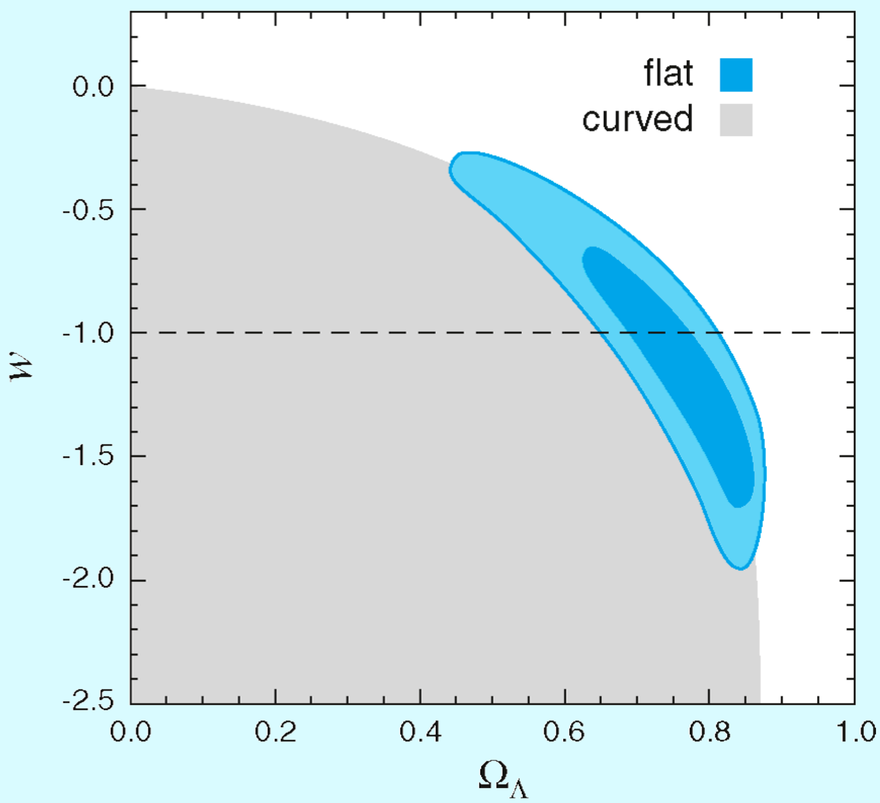
Cosmological parameters

$$D_A = \frac{c}{H_0\left(1+z\right)}\int_0^z \frac{dz'}{E\left(z'\right)}$$

$$E\left(z\right) = \sqrt{\Omega_{m,0}\left(1+z\right)^3 + \Omega_{k,0}\left(1+z\right)^2 + \Omega_{\Lambda,0}\left(1+z\right)^{3(1+w)}}$$

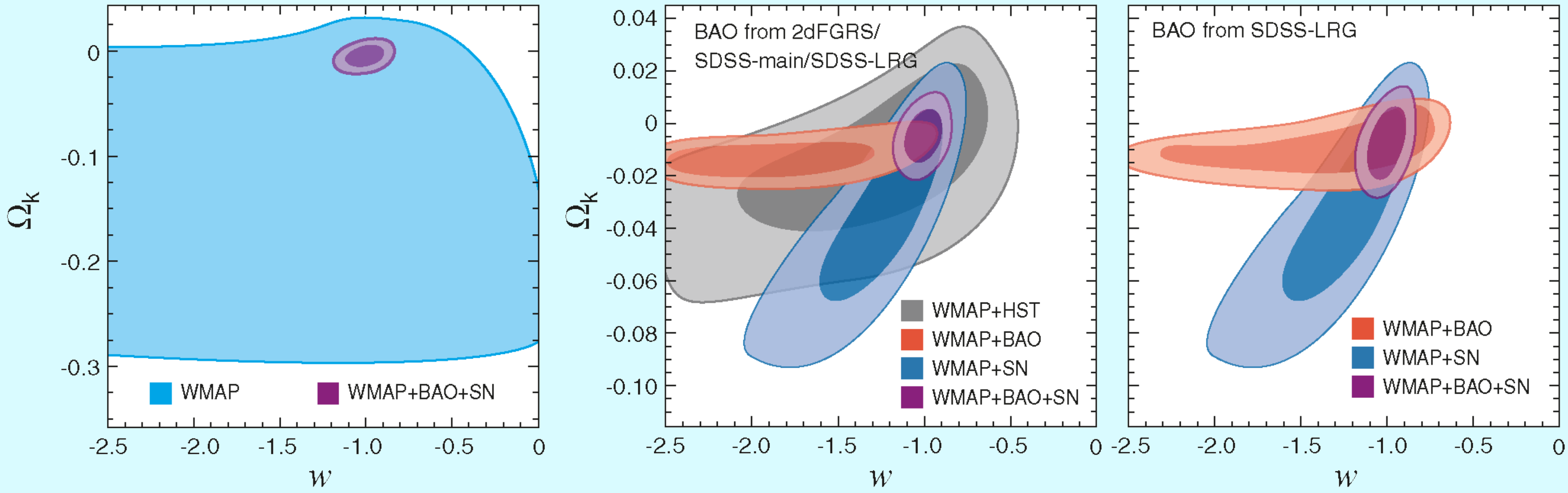
Constraining Cosmological Parameters





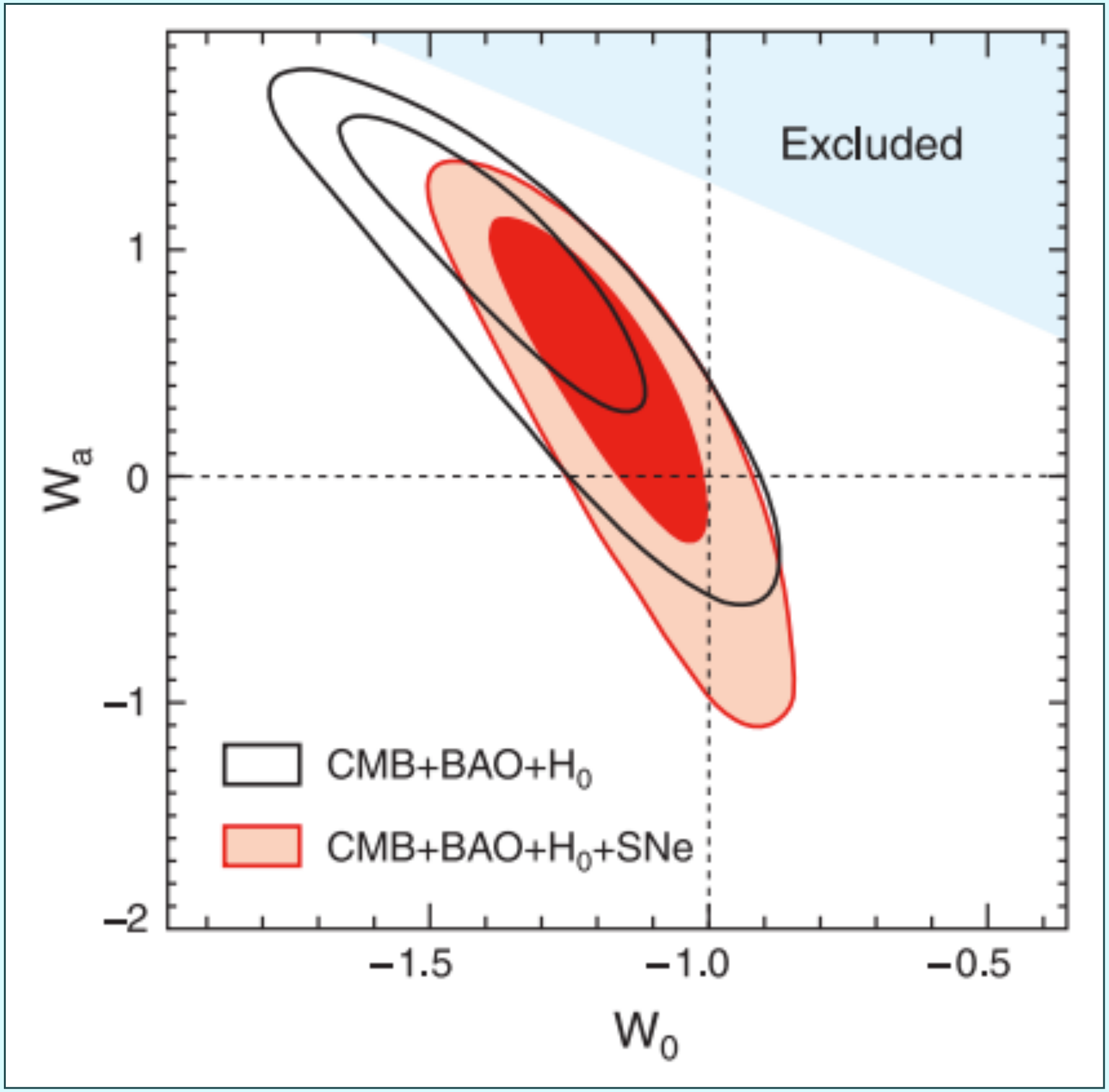
Dunkley et al. (2009)

Constraining Cosmological Parameters



Komatsu et al. (2009)

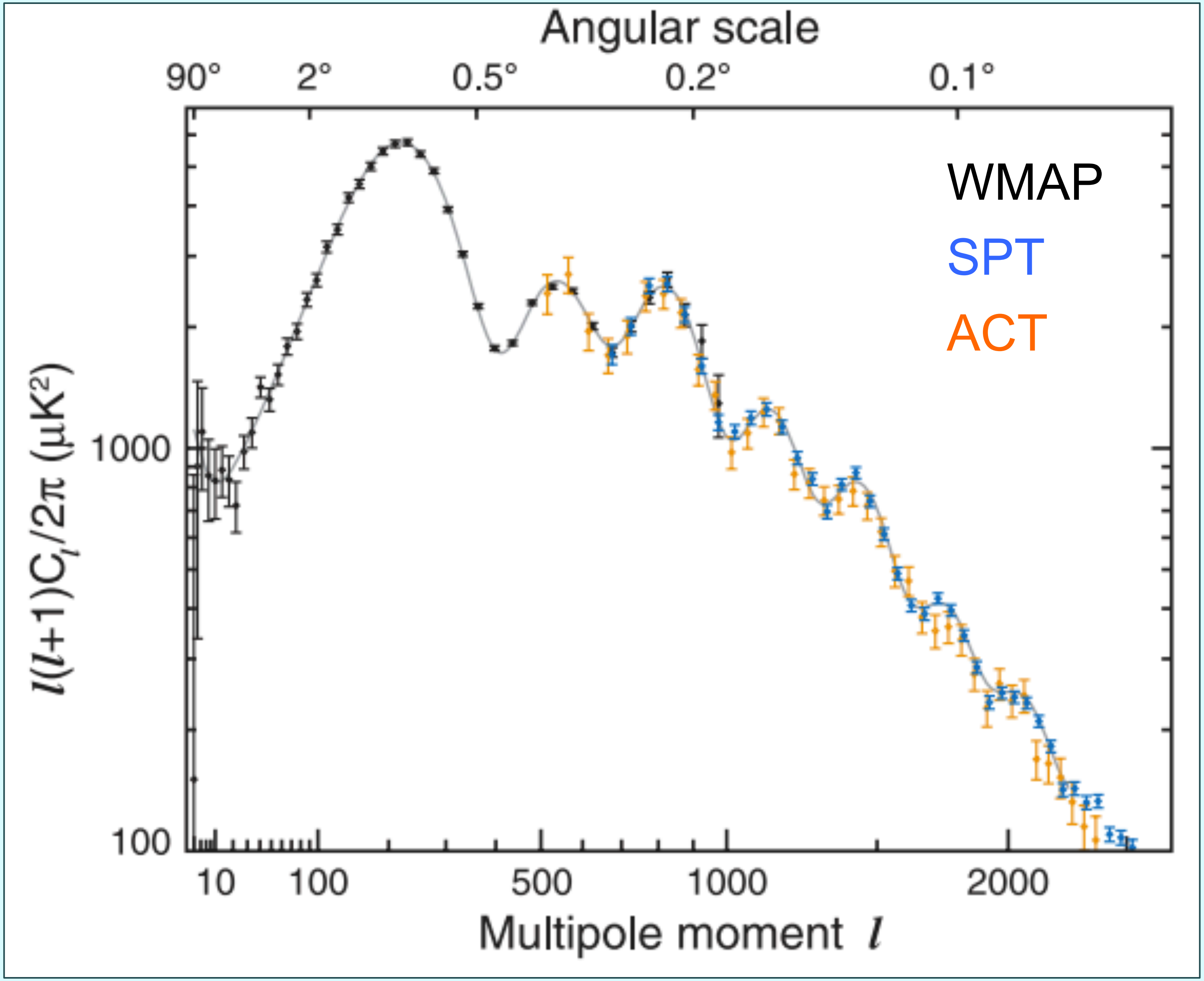
Constraining Cosmological Parameters



$$w = w_0 + w_a a$$

Hinshaw et al. (2013)

Constraining Cosmological Parameters



Hinshaw et al. (2013)