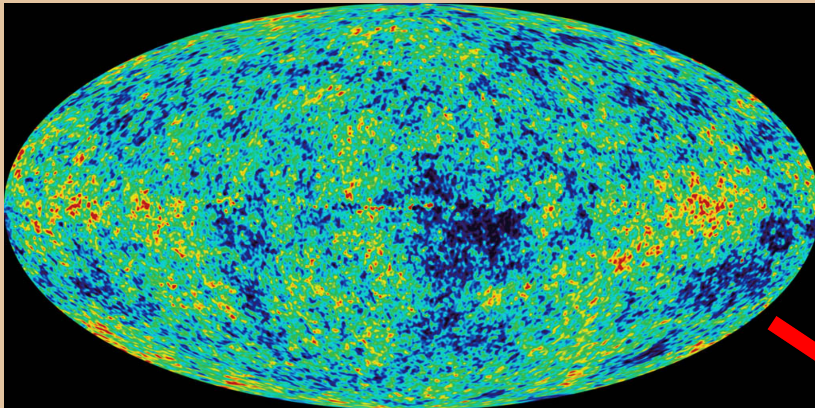
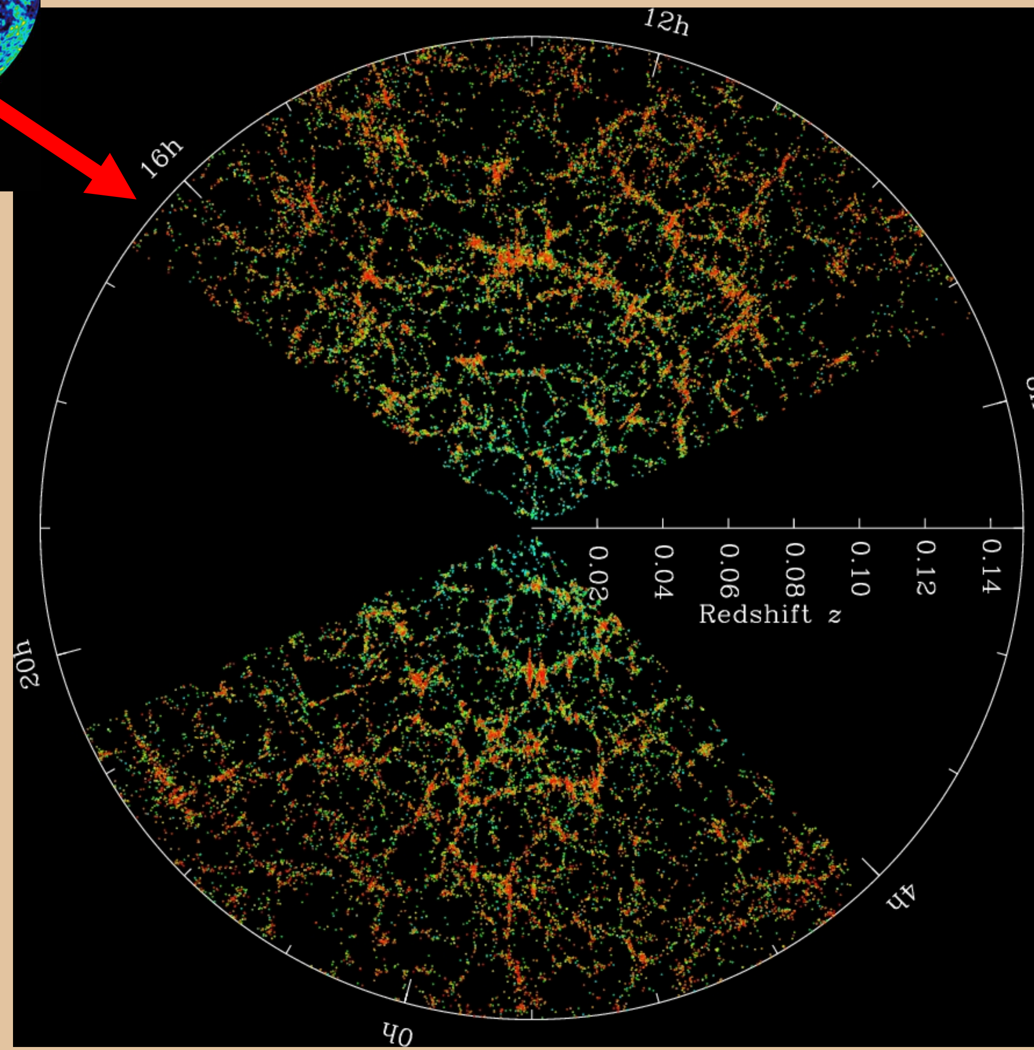
The Growth of Structure





The Growth of Structure

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}}$$

CMB: $\delta \sim 10^{-4}$

Superclusters: $\delta \sim 10$

Clusters: $\delta \sim 10^{2}$

Galaxies: $\delta \sim 10^{4}$

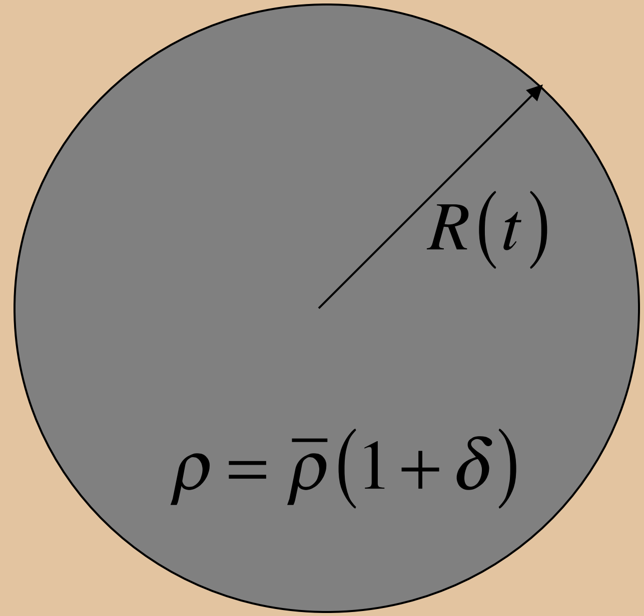
Stars: $\delta \sim 10^{29}$

People: $\delta \sim 10^{30}$

Stellar black hole: $\delta \sim 10^{45}$

The Growth of Structure

Assume small spherical overdensity in a *static* universe.



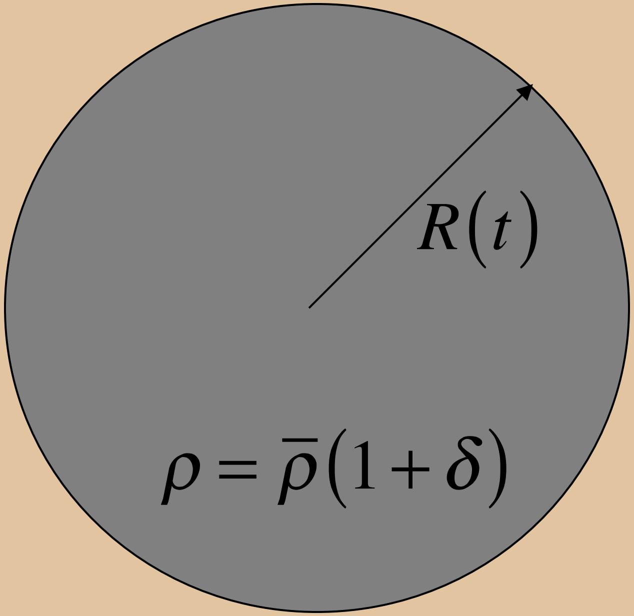
$$\ddot{R} = -\frac{G(\Delta M)}{R^2}$$

$$= -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\bar{\rho}\delta\right)$$

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t)}$$

Two unknowns: $R(t)$ and $\delta(t)$

Conservation of mass



$$M = \frac{4\pi}{3} R(t)^3 \, \bar{\rho} \left[1+\delta(t)\right]$$

$$R(t) = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} \left[1+\delta(t)\right]^{-1/3}$$

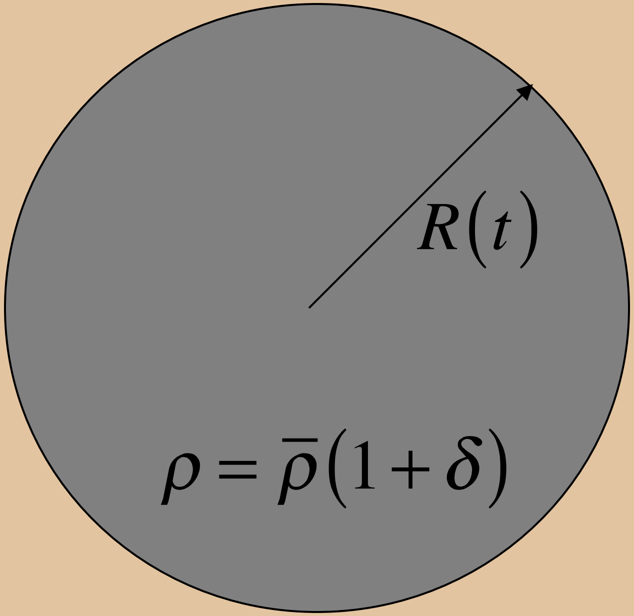
$$R_0 = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} = \text{const}$$

$$\delta \ll 1 \quad \rightarrow \quad R(t) \approx R_0 \left[1 - \frac{1}{3}\delta(t)\right]$$

The Growth of Structure

$$\ddot{R} \approx -\frac{1}{3}R_0\ddot{\delta} \approx -\frac{1}{3}R\ddot{\delta}$$

$$\boxed{\frac{\ddot{R}}{R} \approx -\frac{1}{3}\ddot{\delta}}$$



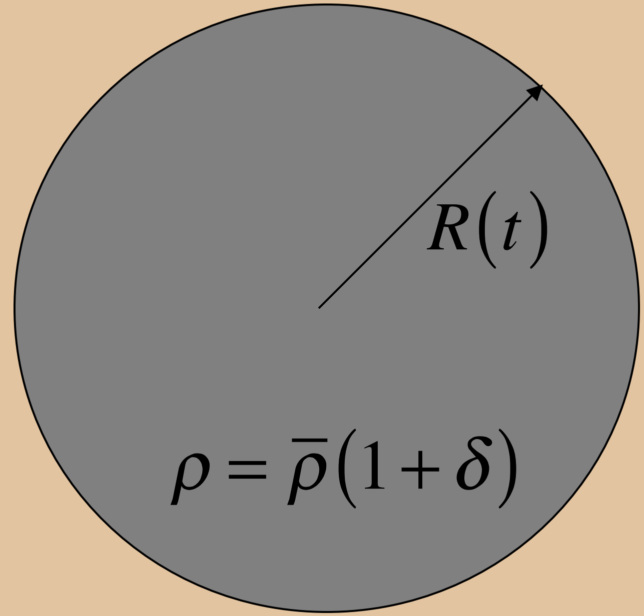
Combine with:

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t)}$$

$$\boxed{\ddot{\delta} = 4\pi G\bar{\rho}\delta}$$

The Growth of Structure

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta$$



General solution:

$$\delta(t) = A_1 e^{t/t_{dyn}} + A_2 e^{-t/t_{dyn}}$$

wher
e

$$t_{dyn} = (4\pi G\bar{\rho})^{-1/2}$$

After a few dynamical times, only the growing mode is significant

The Growth of Structure

$$\delta(t) \sim e^{t}$$

In reality, density perturbations do not grow this fast in the universe because:

- there is some pressure support

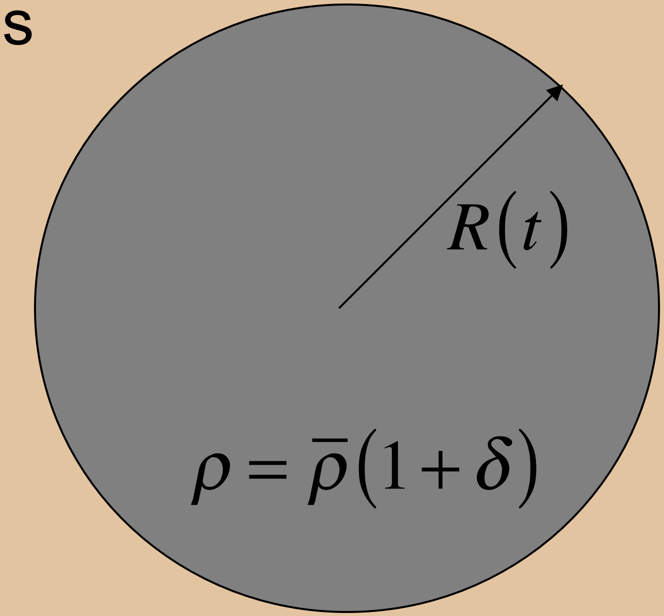
- the universe is not static, but expanding

$$t_{dyn} \sim \left(G\bar{\rho}\right)^{-1/2} = \left(c^2 / G\bar{\varepsilon}\right)^{1/2}$$

$$\bar{\varepsilon} = \frac{3c^2}{8\pi G} H^2 \rightarrow H^{-1} \sim \left(c^2 / G\bar{\varepsilon}\right)^{1/2}$$

The Growth of Structure

Re-do, but now mean density evolves



$$\ddot{R} = -\frac{GM}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}\rho R^3\right)$$

$$= -\frac{4\pi}{3}G\bar{\rho}R - \frac{4\pi}{3}G\left(\bar{\rho}\delta\right)R$$

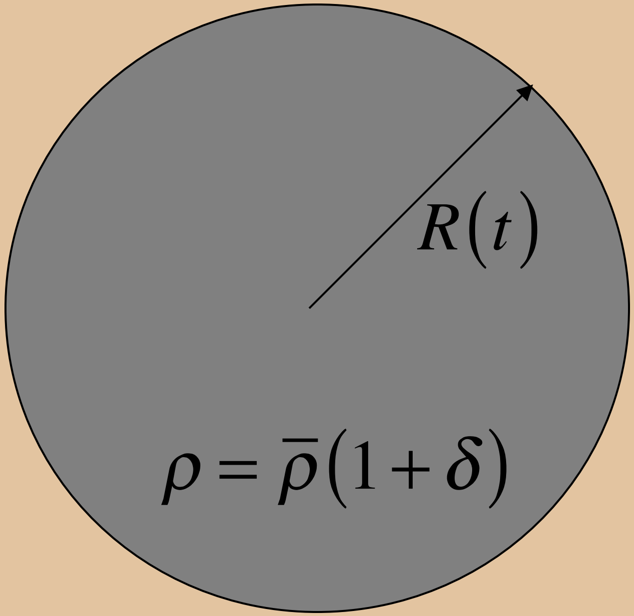
$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta$$

Mass conservation:

$$M = \frac{4\pi}{3}R(t)^3\,\bar{\rho}(t)\left[1+\delta(t)\right] = \text{const}$$

The Growth of Structure

$$R(t) \propto \bar{\rho}(t)^{-1/3}\left[1+\delta(t)\right]^{-1/3}$$



$$\bar{\rho} \propto a^{-3}$$

$$R(t) \propto a(t)\left[1+\delta(t)\right]^{-1/3}$$

$$\approx a(t)\left[1-\frac{1}{3}\delta(t)\right]$$

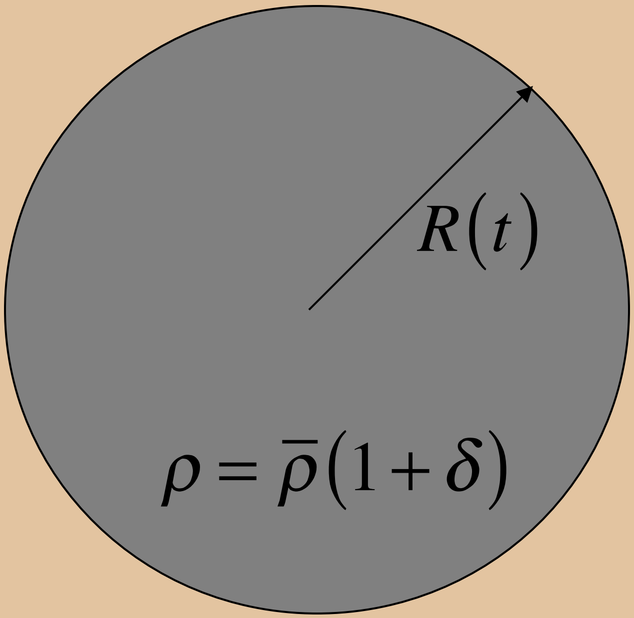
An overdense region will grow slightly less rapidly than the scale factor.

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} \qquad = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta$$

The Growth of Structure

With δ~0, this reduces to:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\bar{\rho}$$



Subtract this from previous equation:

$$-\frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho}\delta$$

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta$$

Has extra term that acts to slow collapse in an expanding universe.

The Growth of Structure

$$\left(\frac{\partial\rho}{\partial t}\right)_r + \rho\nabla_r\cdot\vec{u} = 0$$

Continuity (mass)

$$\left(\frac{\partial\vec{u}}{\partial t}\right)_r + \left(\vec{u}\cdot\nabla_r\right)\vec{u} = -\nabla_r\Phi$$

Euler (momentum)

$$\nabla_r^2\Phi = 4\pi G\rho$$

Poisson (gravity)

Transform to comoving coordinates

$$\vec{r} = a\vec{x}$$

$$\vec{u} = \dot{\vec{r}} = \dot{a}\vec{x} + \vec{v}$$

The Growth of Structure

- Add a small perturbation

$$\rho(\vec{x},t) = \overline{\rho}(t) + \delta(\vec{x},t)\overline{\rho}(t)$$

- Linear approximation: keep first order terms in δ and v
- Rearrange equations to get rid of v
- Subtract off equation for unperturbed case to get δ

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\overline{\rho}\delta$$

The Growth of Structure

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G}{c^2}\bar{\varepsilon}_m \delta$$

δ is the density of *matter* only

$$\delta = \frac{\varepsilon_m - \bar{\varepsilon}_m}{\bar{\varepsilon}_m}$$

$$\Omega_m = \frac{\bar{\varepsilon}_m}{\varepsilon_c} = \frac{8\pi G \bar{\varepsilon}_m}{3c^2 H^2}$$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2 \Omega_m \delta = 0$$

The solution to this equation depends on Ω_m

The Growth of Structure

- Radiation-dominated phase in early universe $\Omega_m \ll 1$

$$H = \frac{1}{2t}$$

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} = 0$$

Solution: $$\delta(t) \approx B_1 + B_2 \ln t$$

Perturbations grow at a logarithmic rate.

The Growth of Structure

- <u>Lambda-dominated phase</u> in late universe $\Omega_m \ll 1$

$$H = H_\Lambda = \text{const}$$

$$\ddot{\delta} + 2H_\Lambda \dot{\delta} = 0$$

Solution: $$\delta(t) \approx C_1 + C_2 e^{-2H_\Lambda t}$$

Perturbations reach a constant amplitude.

The Growth of Structure

- Matter-dominated phase in recent universe $\Omega_m \approx 1$

$$H = \frac{2}{3t}$$

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

Guess: $\delta(t) \approx Dt^n \quad \rightarrow \quad \dot{\delta} = nDt^{n-1} \qquad \ddot{\delta} = n(n-1)Dt^{n-2}$

$$n(n-1)Dt^{n-2} + \frac{4}{3t}nDt^{n-1} - \frac{2}{3t^2}Dt^n = 0$$

$$n(n-1)Dt^{n-2} + \frac{4}{3}nDt^{n-2} - \frac{2}{3}Dt^{n-2} = 0$$

$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = n^2 + \frac{1}{3}n - \frac{2}{3} = 0$$

$$n = \begin{cases} -1 \\ 2/3 \end{cases}$$

The Growth of Structure

- Matter-dominated phase in recent universe $\Omega_m \approx 1$

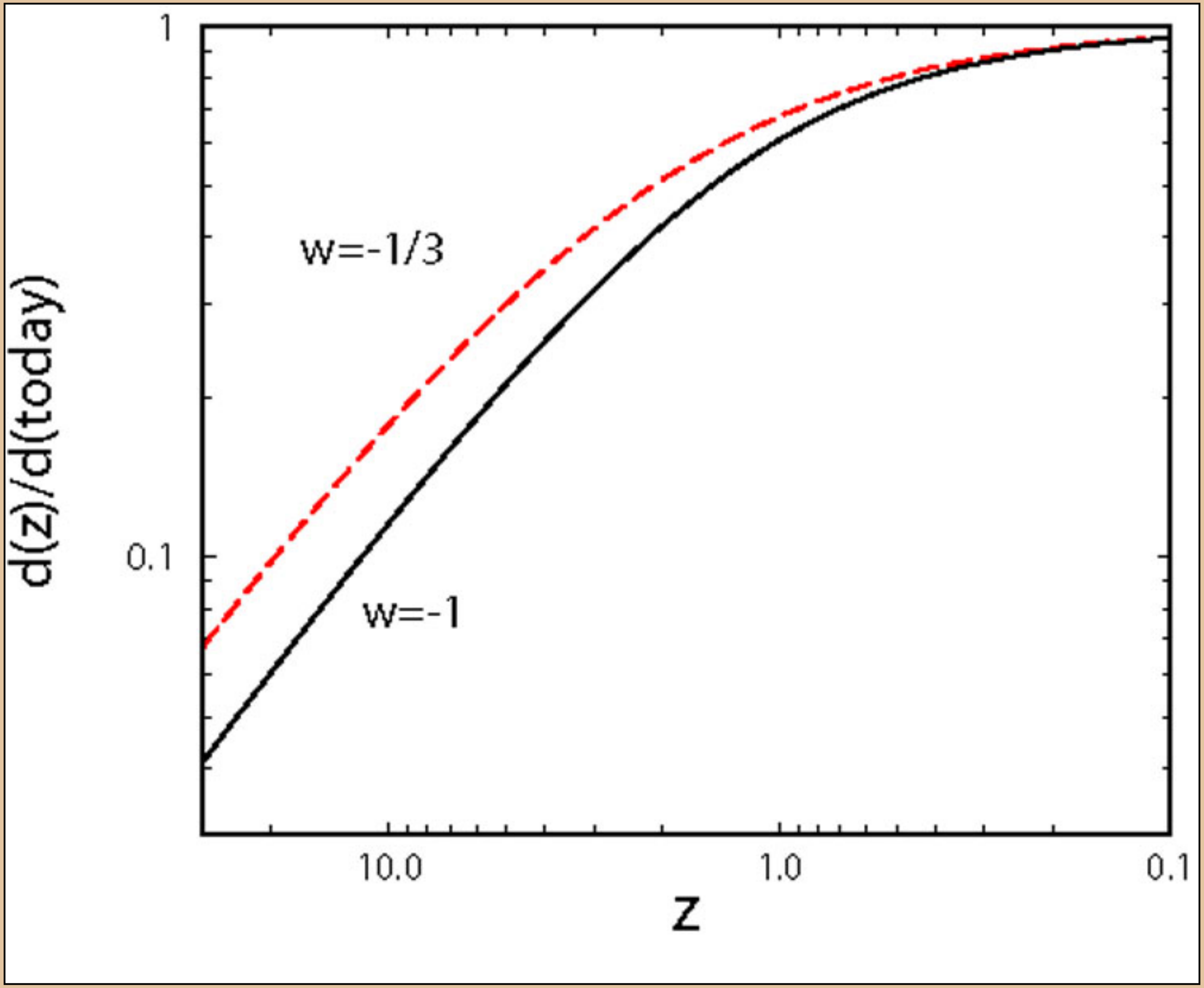
$$H = \frac{2}{3t}$$

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

Solution: $\delta(t) \approx D_1 t^{2/3} + D_2 t^{-1}$

When growing mode dominates, $\delta \propto t^{2/3} \propto a \propto \frac{1}{1+z}$

The Growth of Structure



The Growth of Structure

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\Omega_m\delta = 0$$

None of the terms or derivatives depend on location $\vec{x}$

The solution may thus be written as:

$$\delta(\vec{x},t) = D(t)\tilde{\delta}(\vec{x})$$

$D(t)$ is called the "*growth factor*" and it satisfies the above differential equation. It is also normalized to be equal to unity at t = today.

The Growth of Structure

Since the growth function is normalized to be unity today,

$\tilde{\delta}(\vec{x})$ must be the density at t = today assuming linear

theory. It is the linearly extrapolated density fluctuation.

e.g., in a matter dominated universe: $D(t) = \left(\frac{t}{t_0}\right)^{2/3}$

Large Scale Velocity Field

According to linear theory, the peculiar velocity is:

$$\vec{v}(\vec{x}) = \frac{f(\Omega_m)}{4\pi} \int \delta_m(\vec{x}') \frac{(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^3} d^3x'$$

where:

$$f(\Omega_m) \approx \Omega_m^{0.6}$$

The differential form of this equation is:

$$\vec{\nabla} \cdot \vec{v}(\vec{x}) = -f(\Omega_m)\delta_m(\vec{x})$$

Large Scale Velocity Field

We can measure the radial peculiar velocity of a galaxy by measuring its redshift and a redshift-independent distance.

$$v_r = cz - H_0 d$$

By comparing the observed velocity field to the observed density field, we can constrain Ω. There are two main approaches that have been used:

•velocity-velocity comparison

•density-density comparison

Large Scale Velocity Field

Velocity-Velocity

•Measure the density field from a galaxy redshift survey, smoothing on some (small) scale.

•Use linear theory to predict the full 3D velocity field.

•Predict radial velocities for all the galaxies.

•Compare these predictions to the actual measured galaxy velocities.

•The slope of the predicted vs. observed relation gives us $f(\Omega_m)$.

Large Scale Velocity Field

Density-Density

•Measure the radial velocity field from a galaxy redshift survey, smoothing on some (large) scale.

•Integrate this radially to get the potential field and compute the gradient of the potential field to get the full 3D velocity field. Then use linear theory to predict the density field.

•Compare this prediction to the actual measured galaxy density field.

•The slope of the predicted vs. observed relation gives us $f(\Omega_m)$.

Large Scale Velocity Field

Density-Density

$$\Phi(\vec{r}) = -\int_0^r v_r(r', \theta, \varphi)\, dr'$$

$$\vec{v}(\vec{r}) = -\nabla_r \Phi(\vec{r})$$

$$\delta_m(\vec{r}) = -f(\Omega_m)^{-1} \nabla_r \cdot \vec{v}(\vec{r})$$

Large Scale Velocity Field

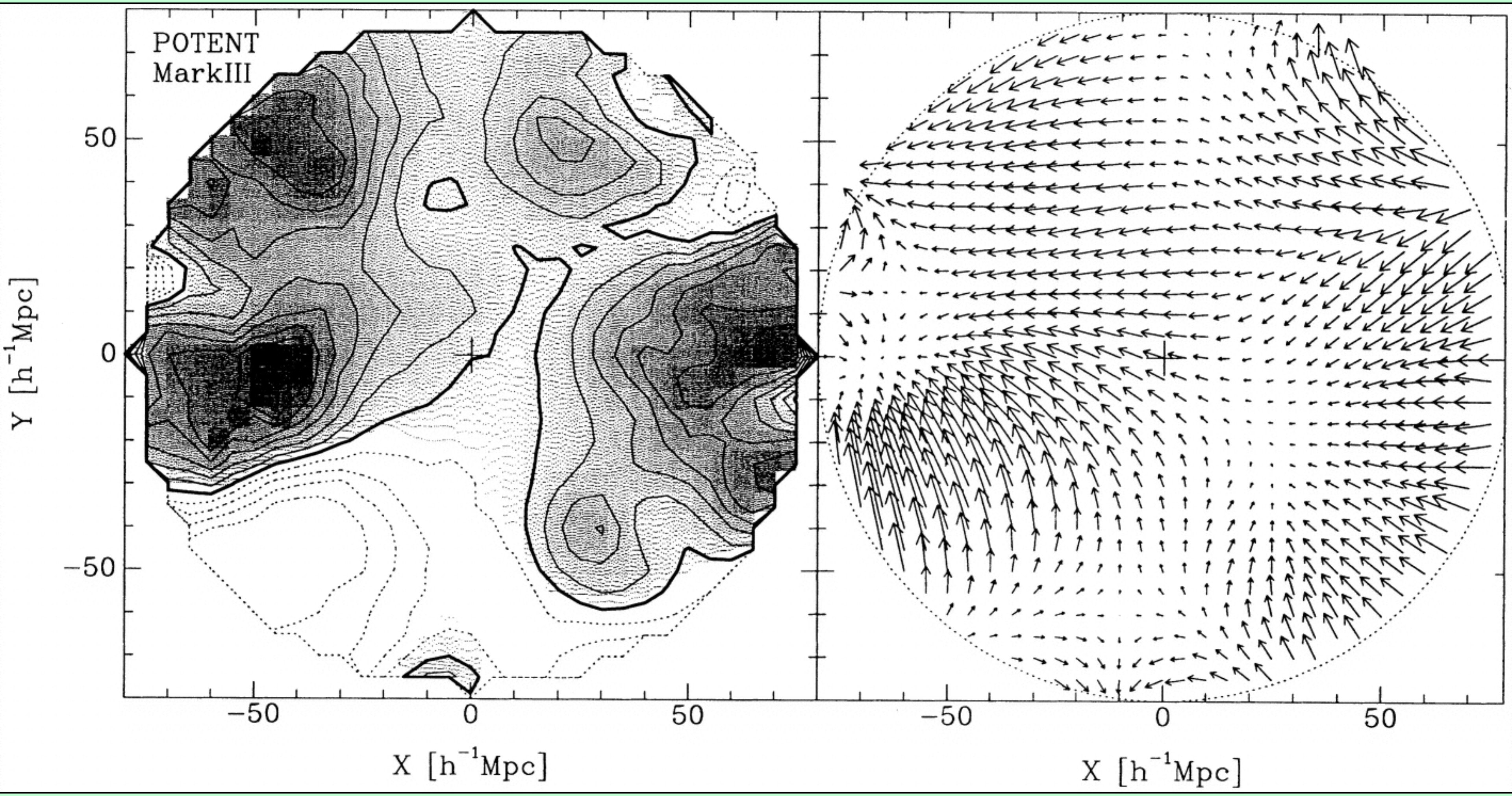
We only actually measure the *galaxy* density field, which on large scales is related to the mass density via a linear bias factor:

$$\delta_g = b\delta_m$$

So these methods actually constrain the quantity:

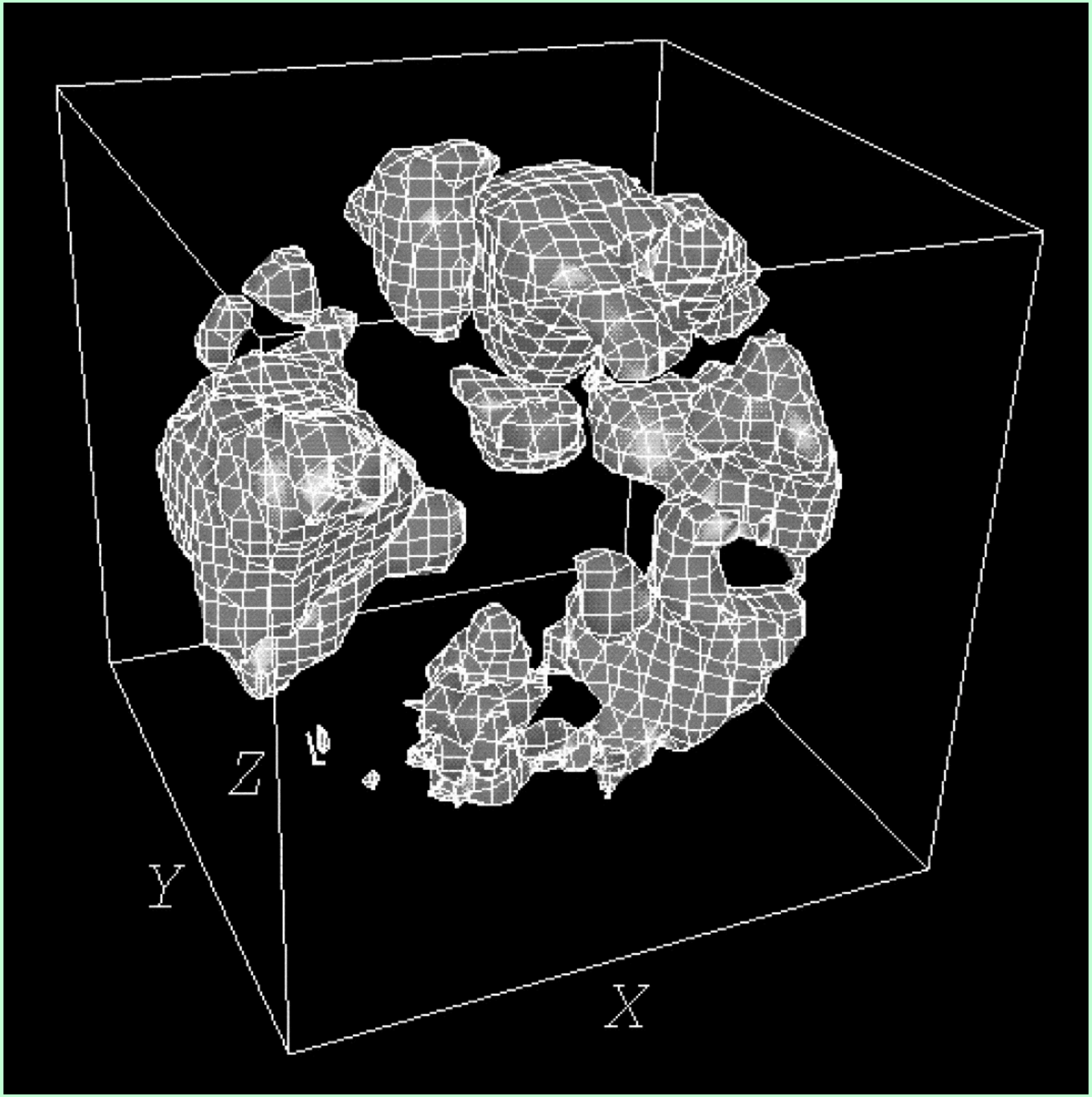
$$\beta = \frac{f\left(\Omega_m\right)}{b} \approx \frac{\Omega_m^{0.6}}{b}$$

Large Scale Velocity Field



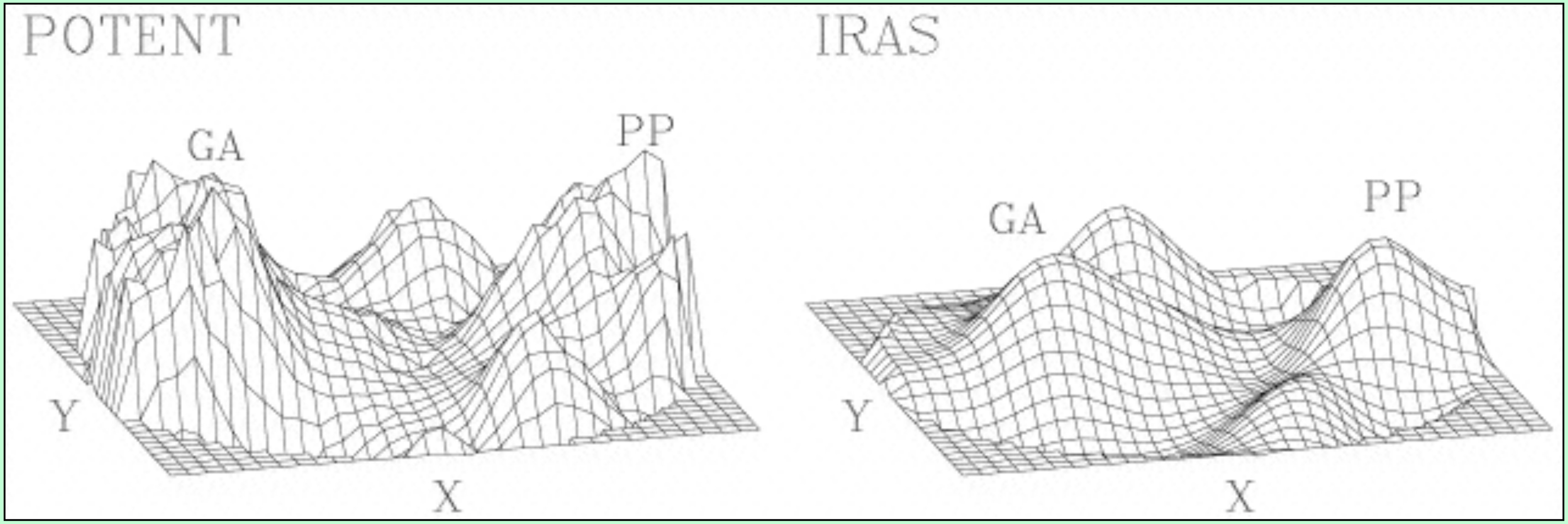
Dekel et al. (1999)

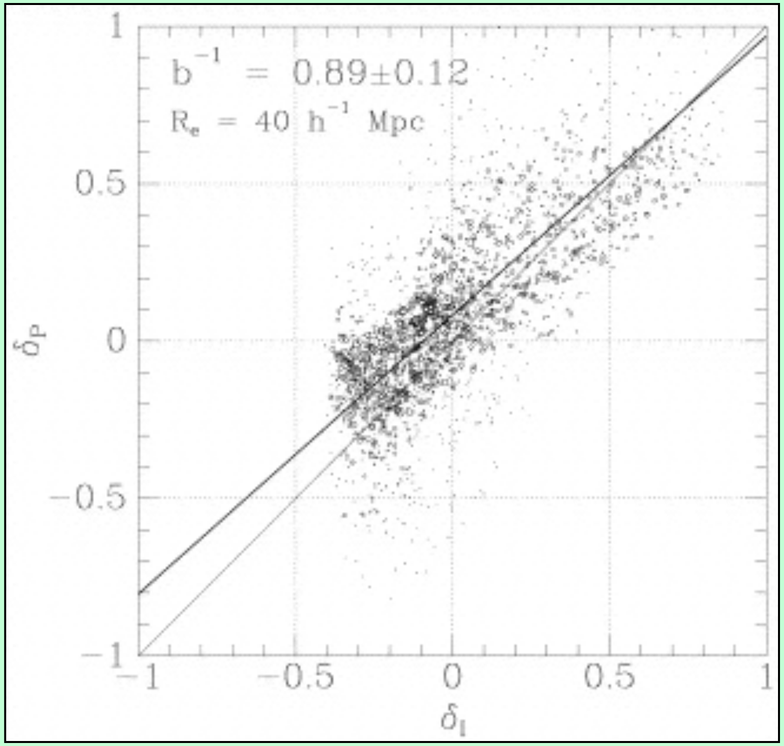
Large Scale Velocity Field



Dekel et al. (1999)

Large Scale Velocity Field



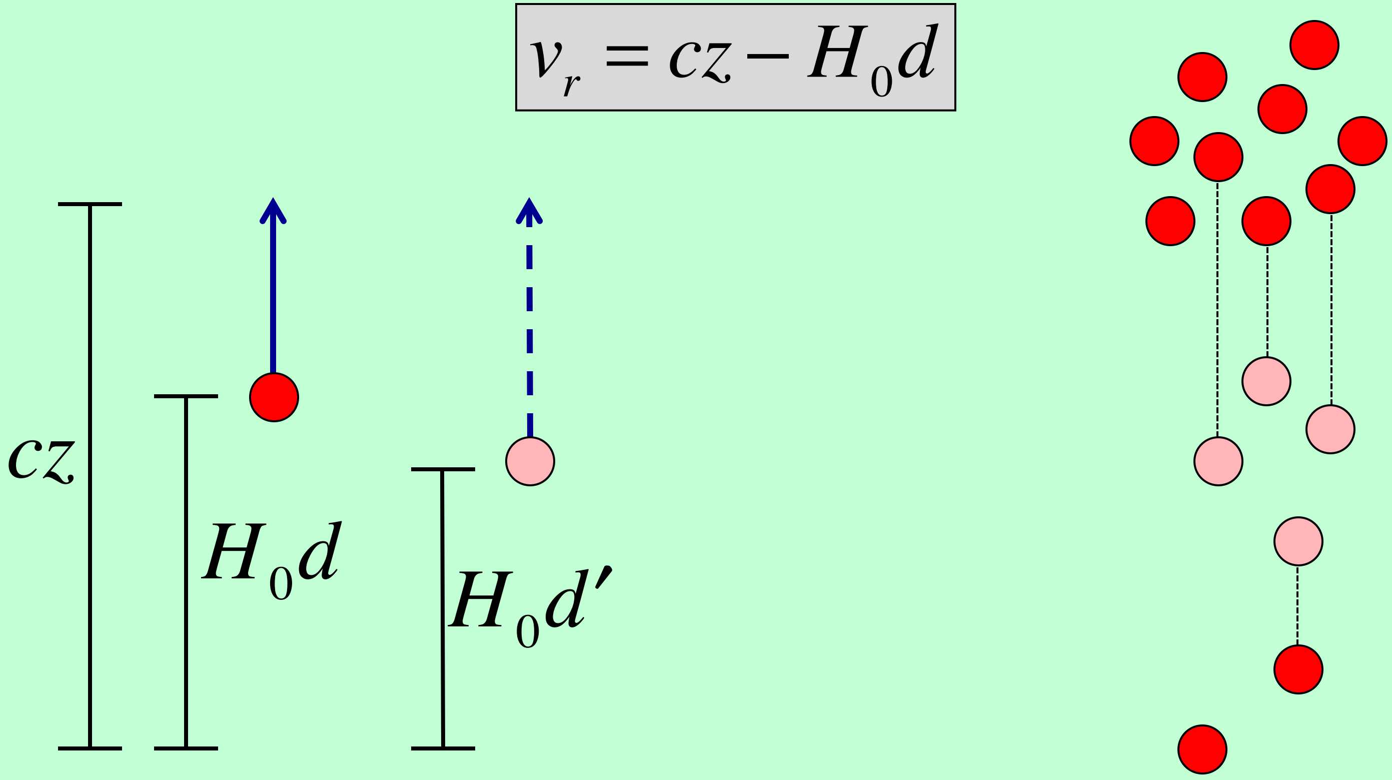


Sigad et al. (1998)

Many systematic errors!

For example, homogeneous and inhomogeneous Malmquist bias, which is caused by anisotropic scattering of galaxy positions due to large distance errors.

$$v_r = cz - H_0 d$$

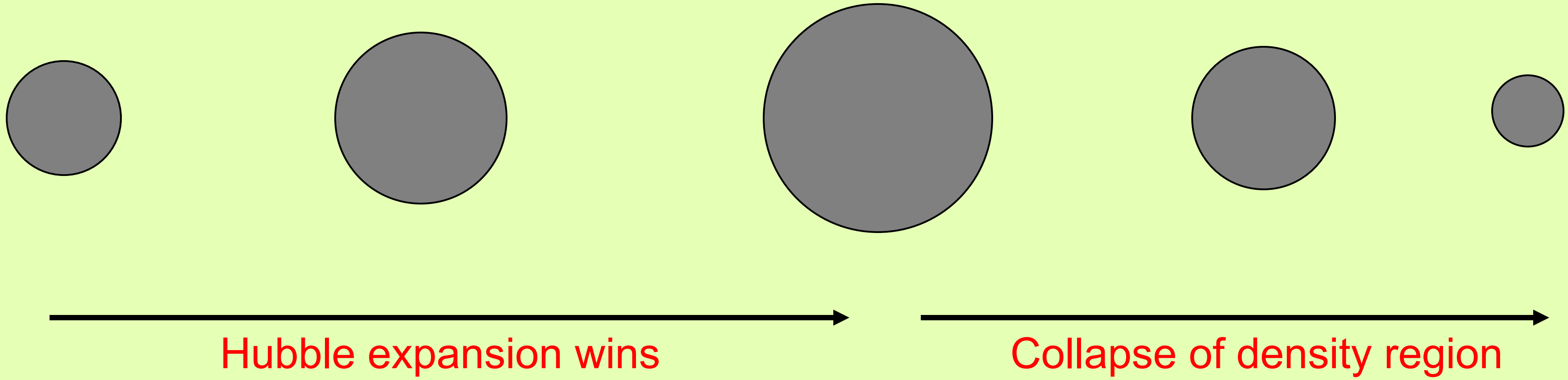


The Spherical Collapse model

Reminder: these solutions are for linear theory only!

Once δ grows to ~1, they do not apply.

We need different solutions to describe the collapse of density fluctuations.



The Spherical Collapse model

The Friedmann equation applies to a small density perturbation, in addition to the whole universe.

In a matter-dominated universe, has a parametric solution:

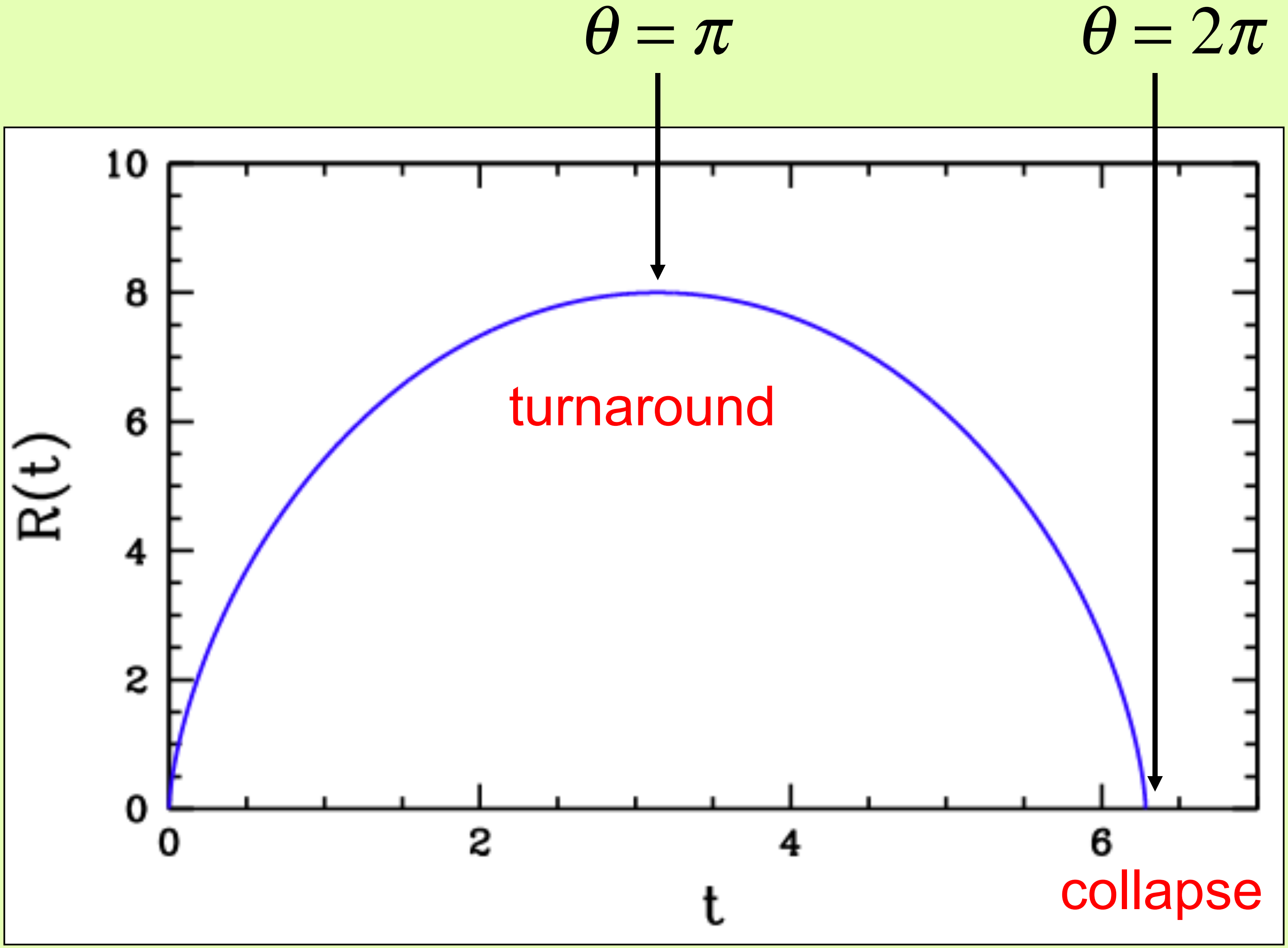
$$\ddot{R} = -\frac{GM}{R^2}$$

$$R = A\left[1 - \cos(\theta)\right]$$
$$t = B\left[\theta - \sin(\theta)\right]$$

(the "cycloid" solution)

Where, $A^3 = GMB^3$

The Spherical Collapse model



The Spherical Collapse model

Expand and only keep low order terms:

$$R = A\left[1 - \cos(\theta)\right]$$

$$t = B\left[\theta - \sin(\theta)\right]$$

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \ldots$$

$$\sin(\theta) \approx \theta - \frac{\theta^3}{6} + \ldots$$

$$R \approx A\left[\frac{\theta^2}{2} - \frac{\theta^4}{24}\right] \quad = \frac{A}{2}\theta^2\left[1 - \frac{\theta^2}{12}\right]$$

$$t \approx B\left[\frac{\theta^3}{6}\right] \quad \rightarrow \quad \theta \approx \left(\frac{6t}{B}\right)^{1/3}$$

$$\rightarrow \quad \mathrm{R} \approx \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1 - \frac{1}{12}\left(\frac{6t}{B}\right)^{2/3}\right]$$

The Spherical Collapse model

$$R(t) \approx \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1 - \frac{1}{12}\left(\frac{6t}{B}\right)^{2/3}\right]$$

Compare this to our previous linear theory result:

$$R(t) \approx a(t)\left[1 - \frac{1}{3}\delta(t)\right]$$

where:

$$a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3}$$ and: $$\delta(t) \propto t^{2/3}$$

The cycloid solution at small t agrees with linear theory.

The Spherical Collapse model

Turnaround

The sphere breaks away from general expansion and reaches a maximum radius at $\theta=\pi$. At this point, linear theory predicts that the density contrast is $\delta_{lin}=1.06$

Collapse

The sphere collapses to a singularity at $\theta=2\pi$. This occurs when $\delta_{lin}=1.69$

Virialization

Complete collapse never occurs in practice because the kinetic energy of collapse is converted into random motions. When the sphere has collapsed to half its maximum size, its kinetic energy is $K=-0.5U$, where U is the potential energy. This is the condition for equilibrium according to the virial theorem. This occurs at $\theta=3\pi/2$ when the density contrast is $\delta_{lin}=1.58$

The Spherical Collapse model

If virialization occurs at 3π/2: $$1+\delta_{\rm vir} \equiv \Delta_{\rm vir} = \frac{\rho}{\bar{\rho}} \approx 147$$

If virialization occurs at 2π: $$1+\delta_{\rm vir} \equiv \Delta_{\rm vir} = \frac{\rho}{\bar{\rho}} \approx 178$$

More generally: $$\Delta_{\rm vir} \approx 178\Omega_m^{-0.7}$$

Even more generally (for flat matter + dark energy models):

$$\Delta_{\rm vir} \approx \left[18\pi^2 + 82\left(\Omega_m - 1\right) - 39\left(\Omega_m - 1\right)^2\right]\Omega_m^{-1}$$

Bryan & Norman (1998)