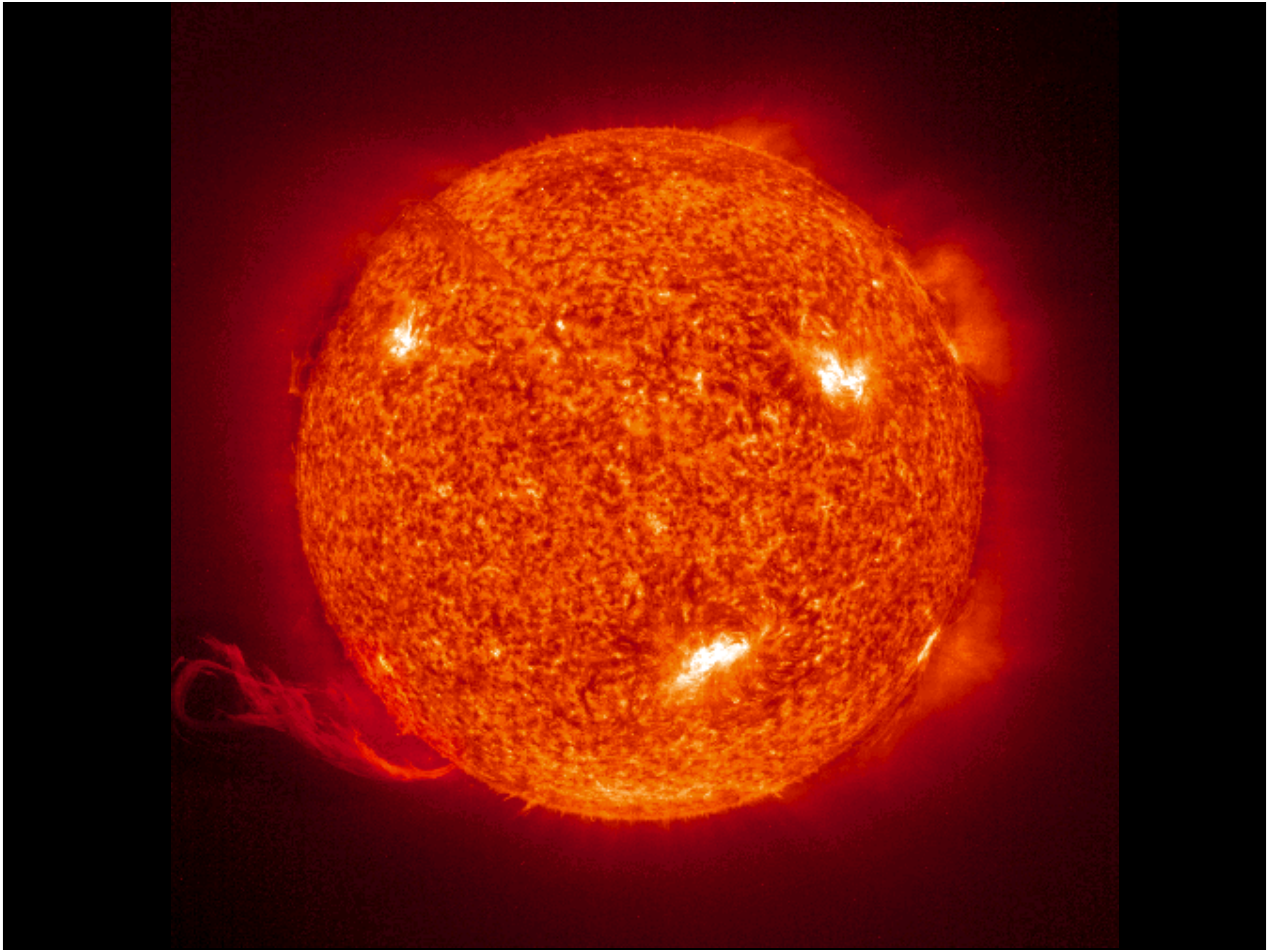
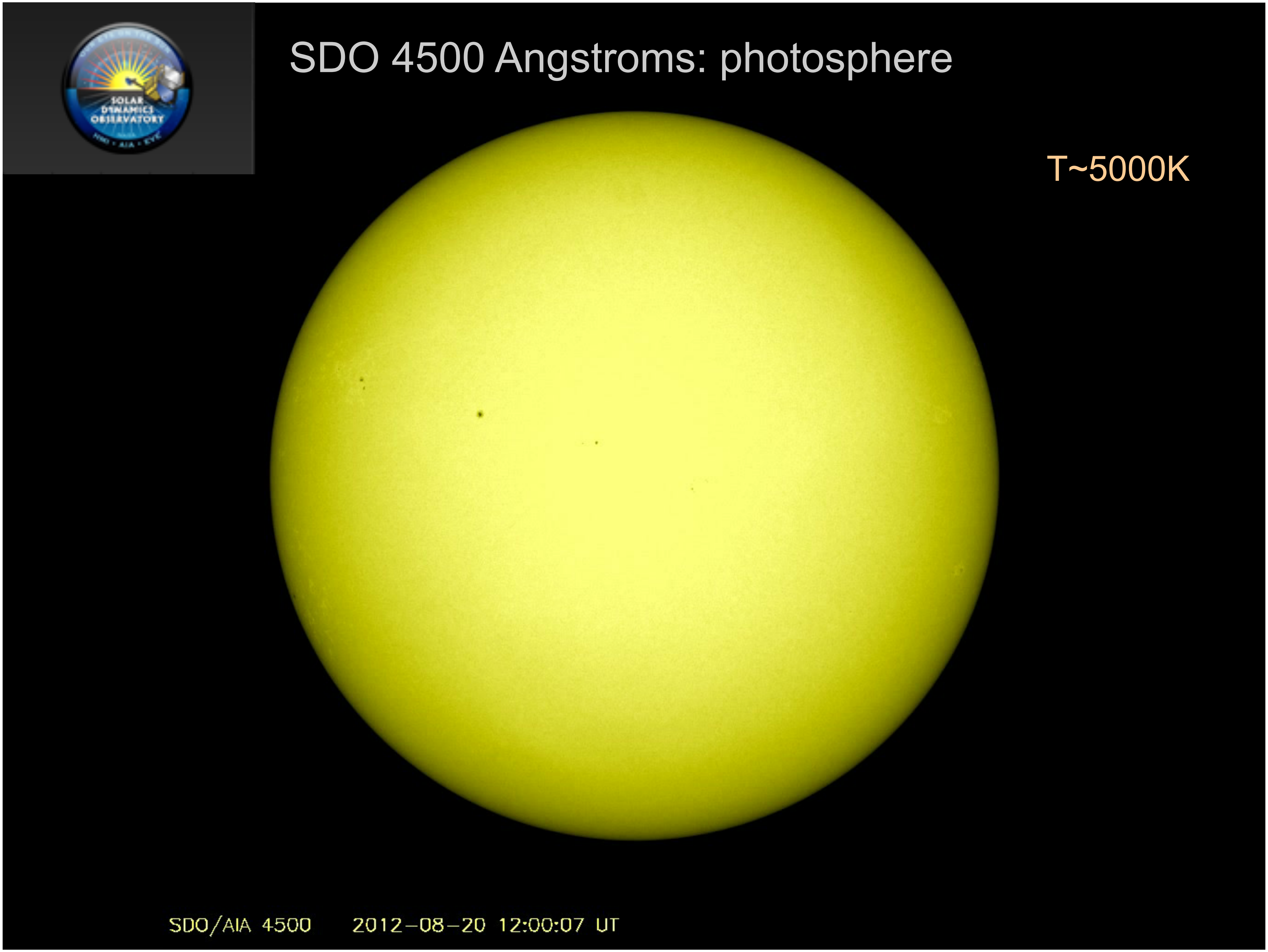
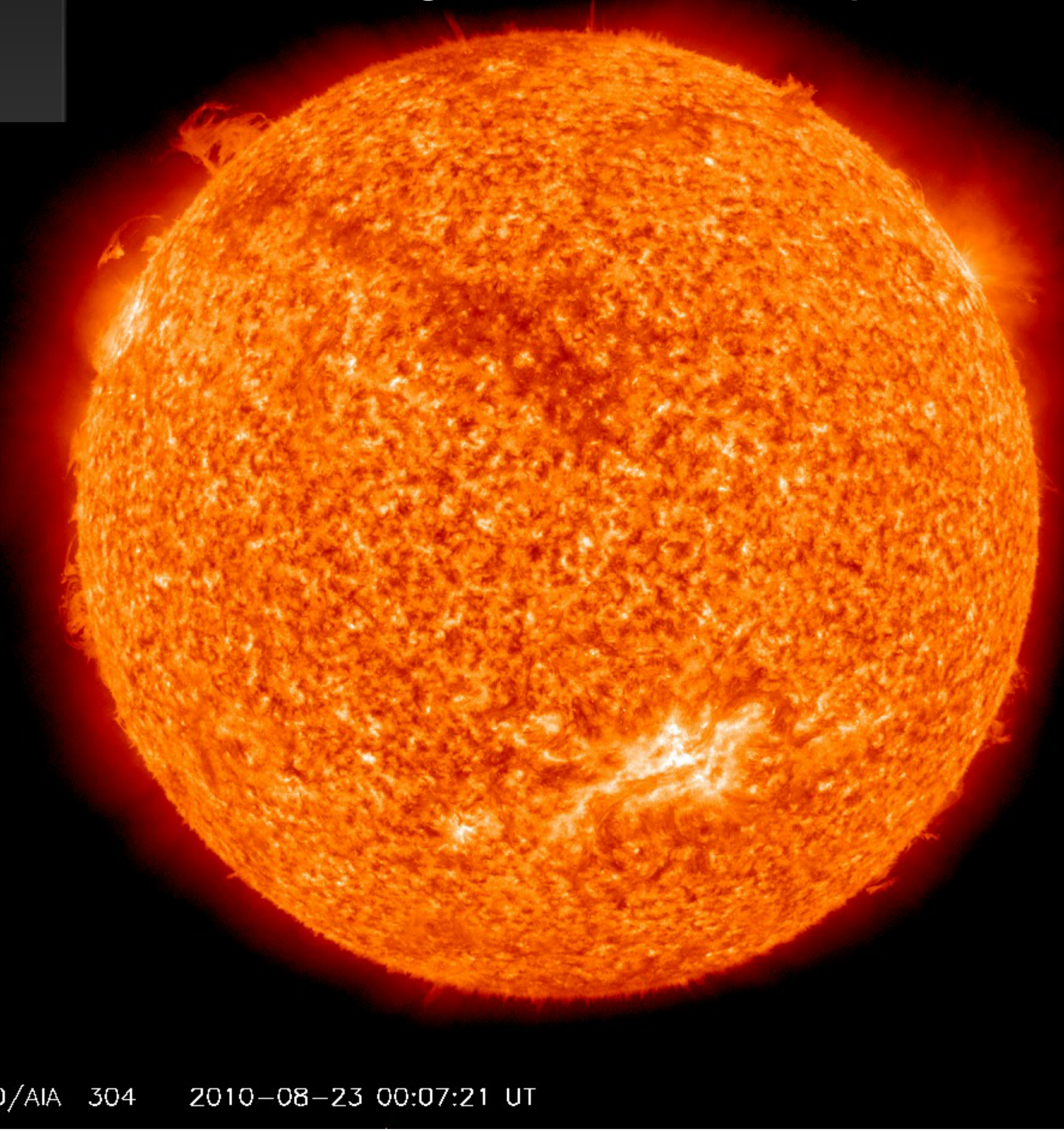


SDO 4500 Angstroms: photosphere



SDO 304 Angstroms: chromosphere

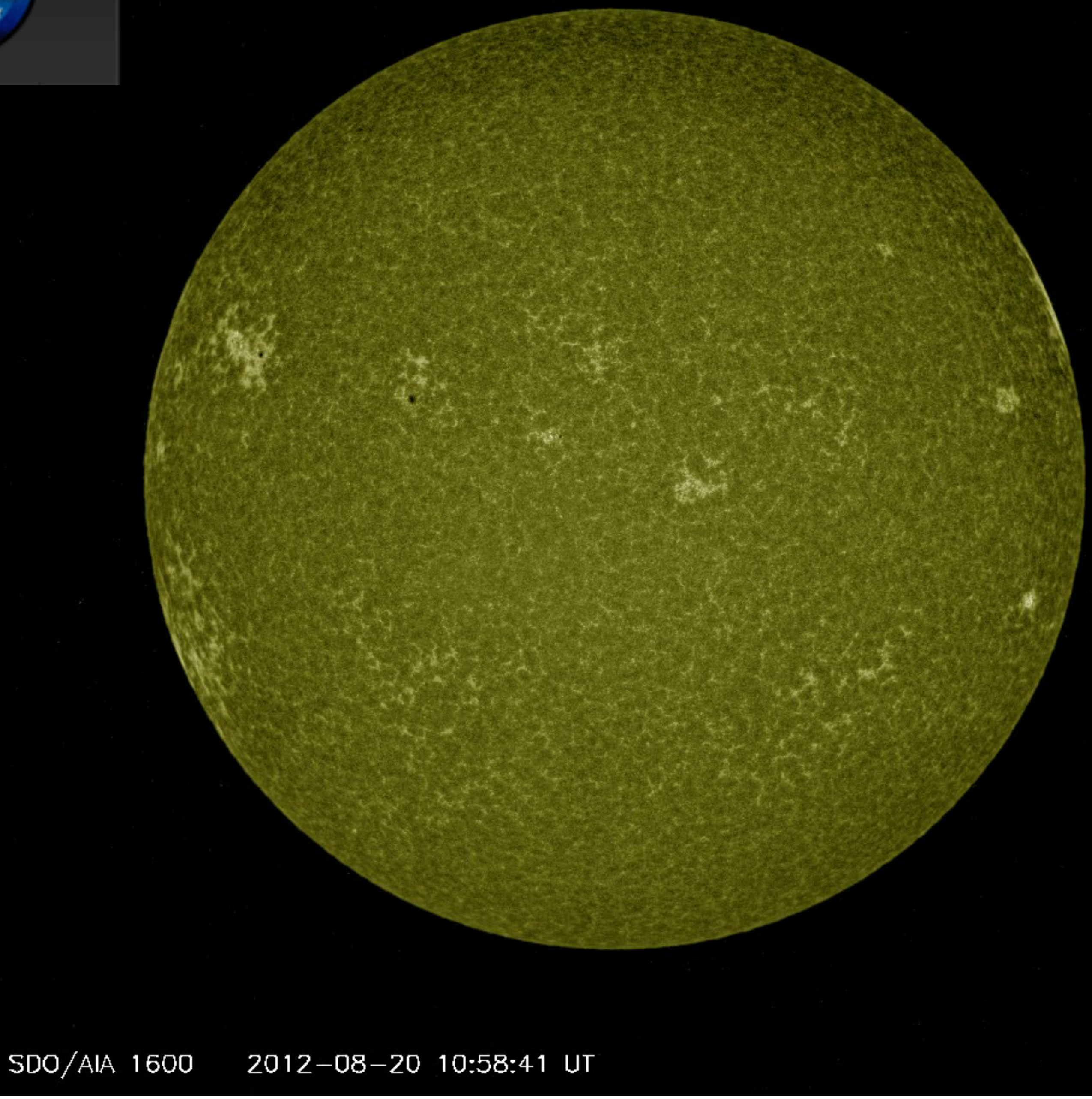
T~5x10^{4}K



SDO/AIA 304 2010-08-23 00:07:21 UT

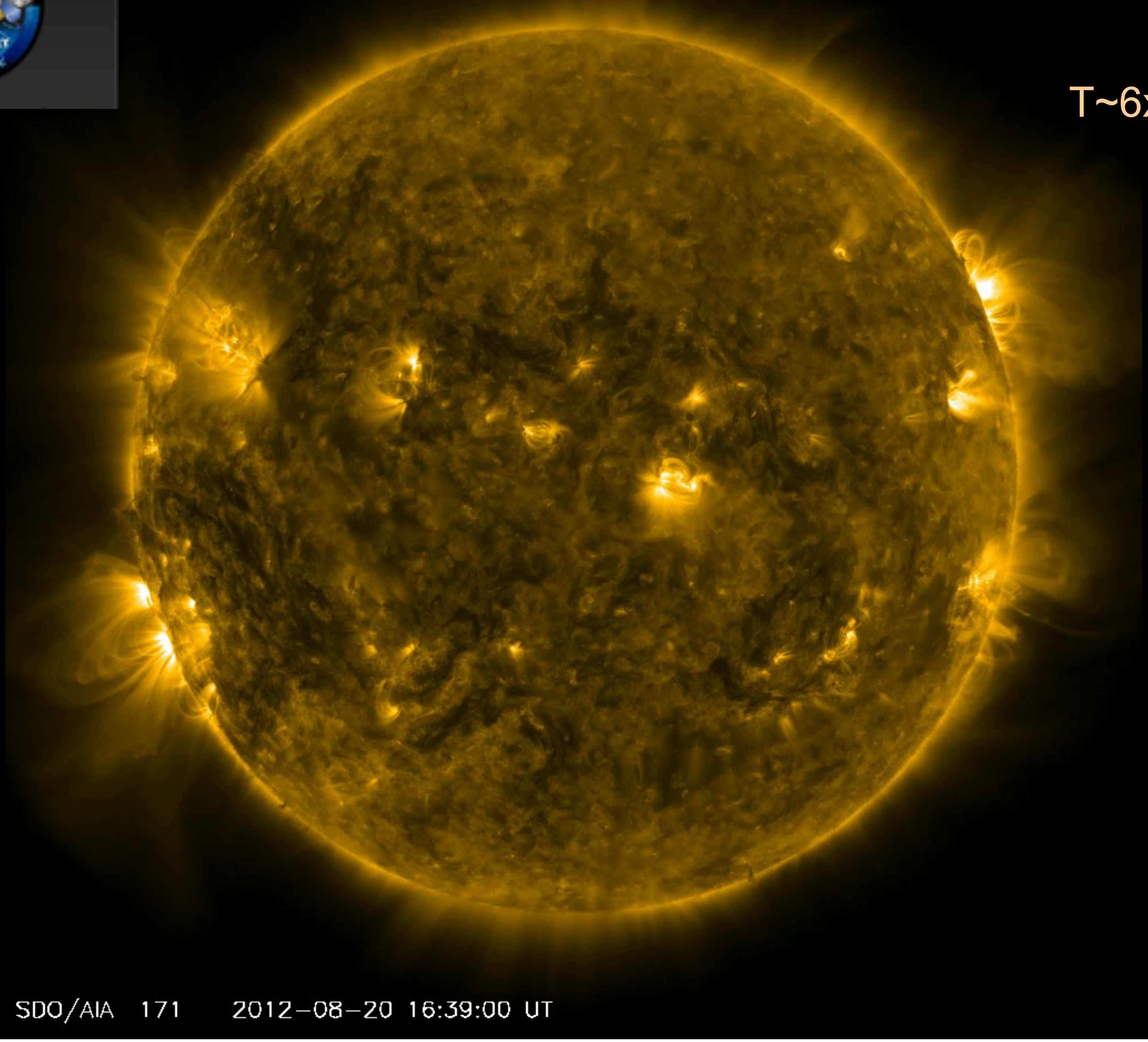
SDO 1600 Angstroms: upper photosphere

T~10^5K



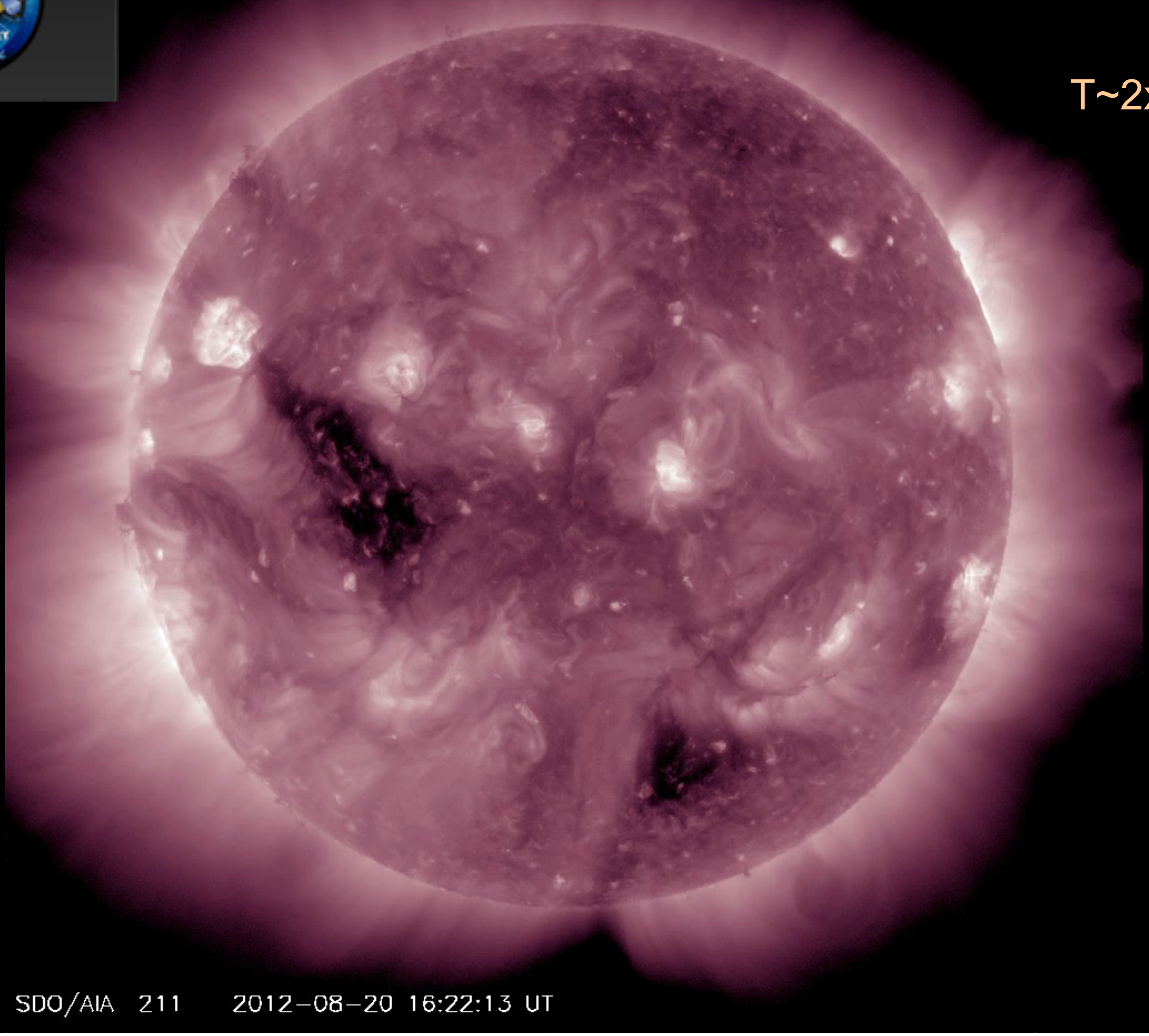
SDO 171 Angstroms: quiet corona

$T \sim 6 x 10^5 K$



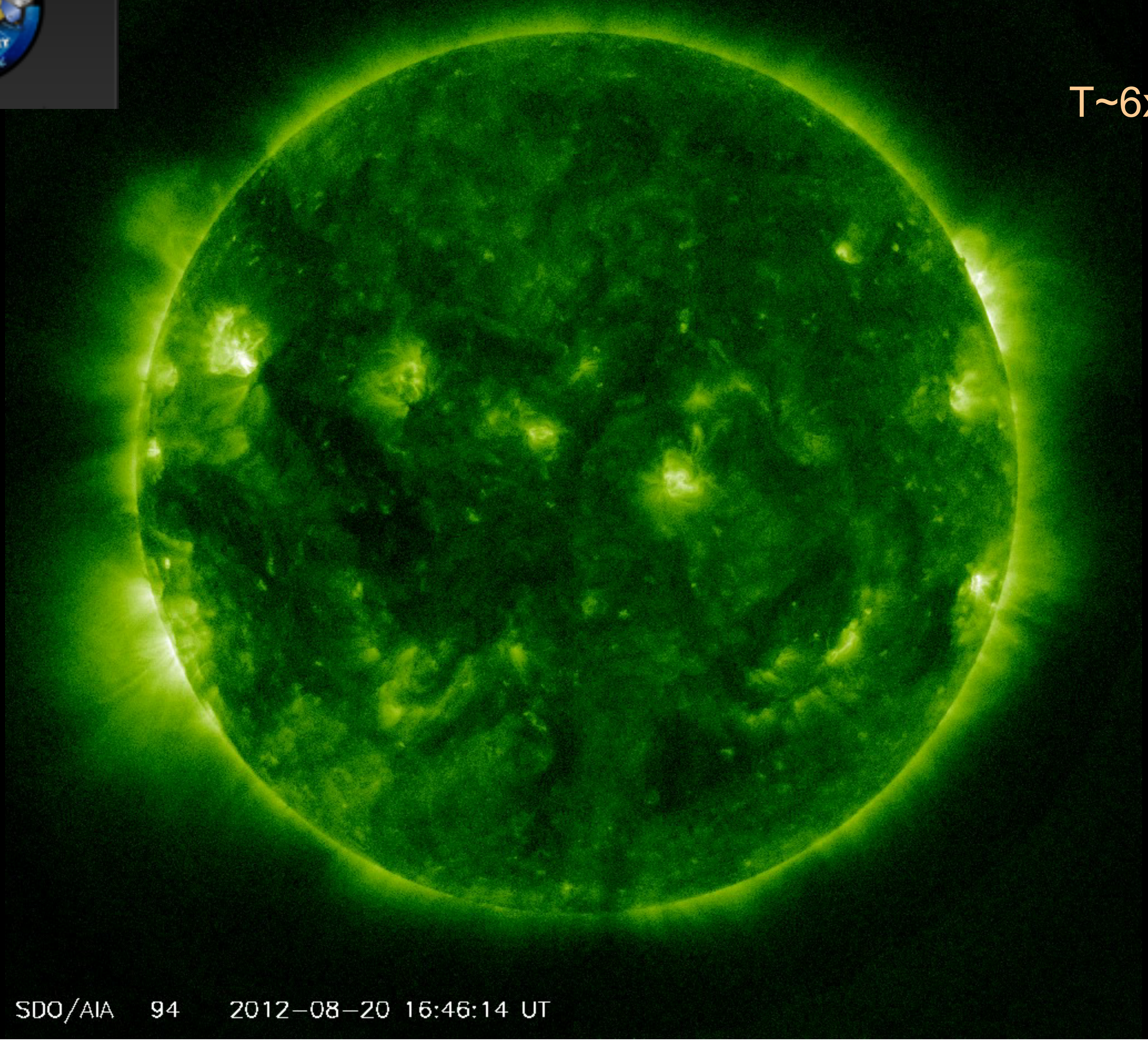
SDO 211 Angstroms: active corona

T~2×10^6K

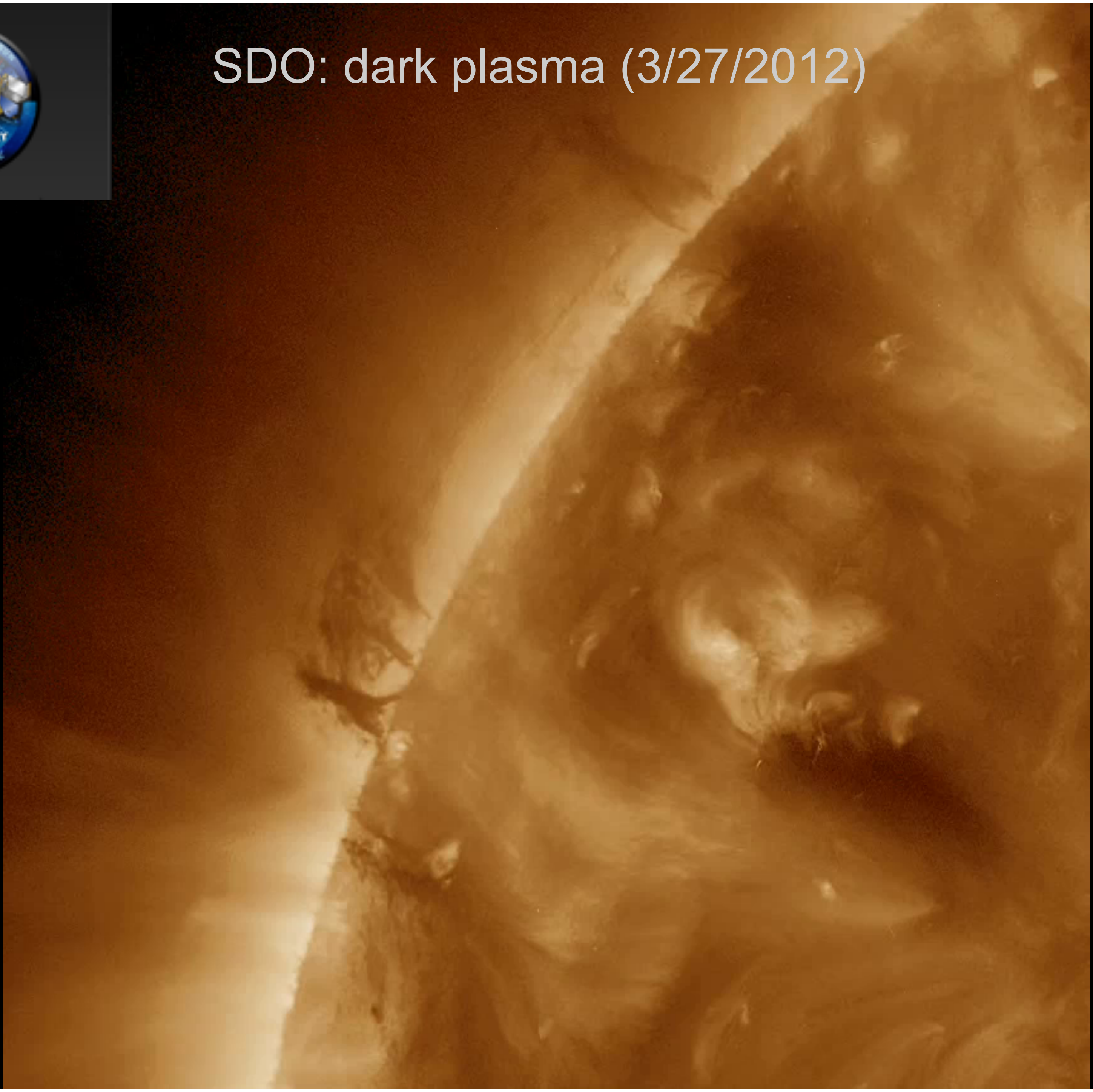


SDO 94 Angstroms: flaring regions

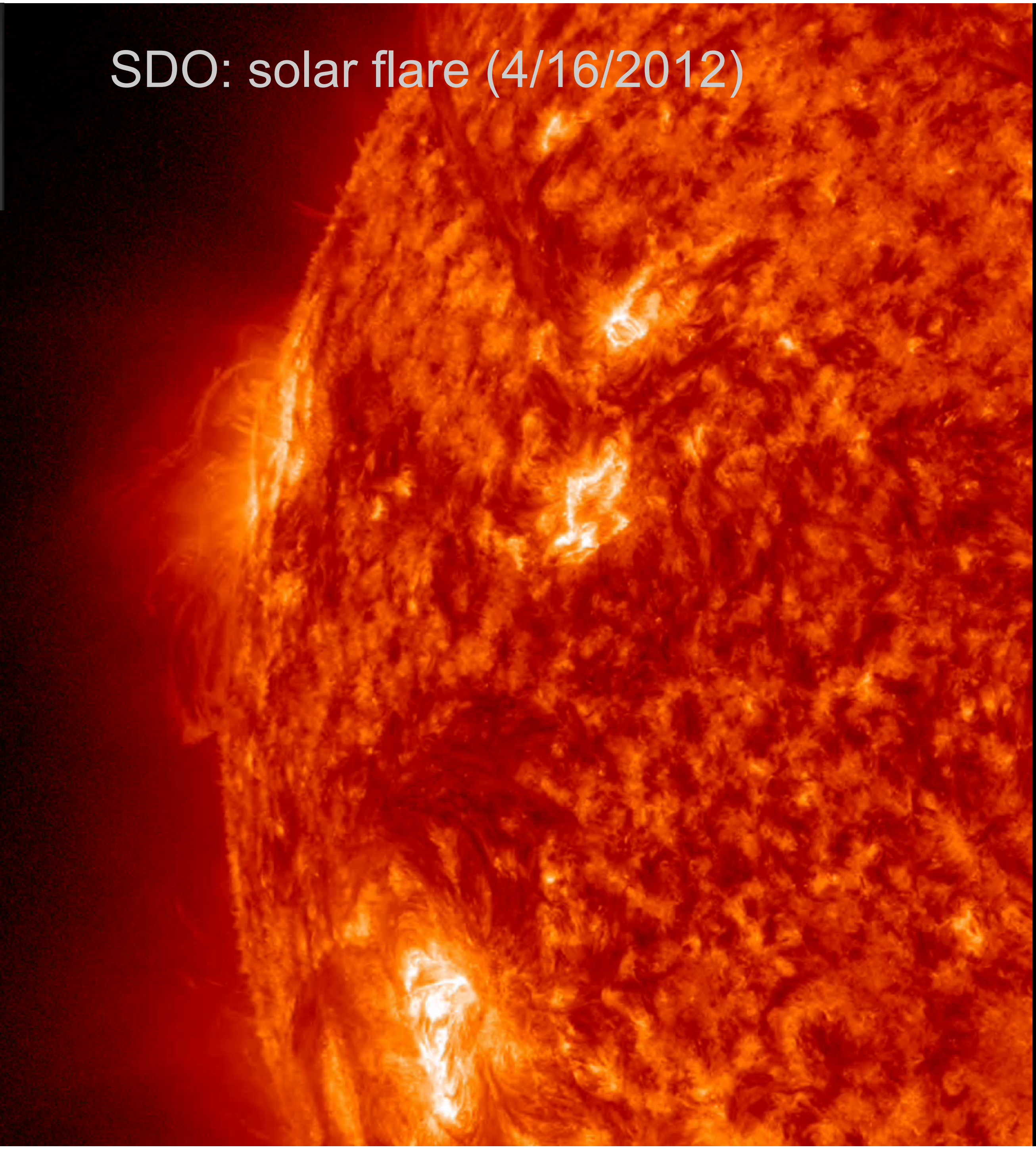
$T \sim 6 \times 10^6 K$



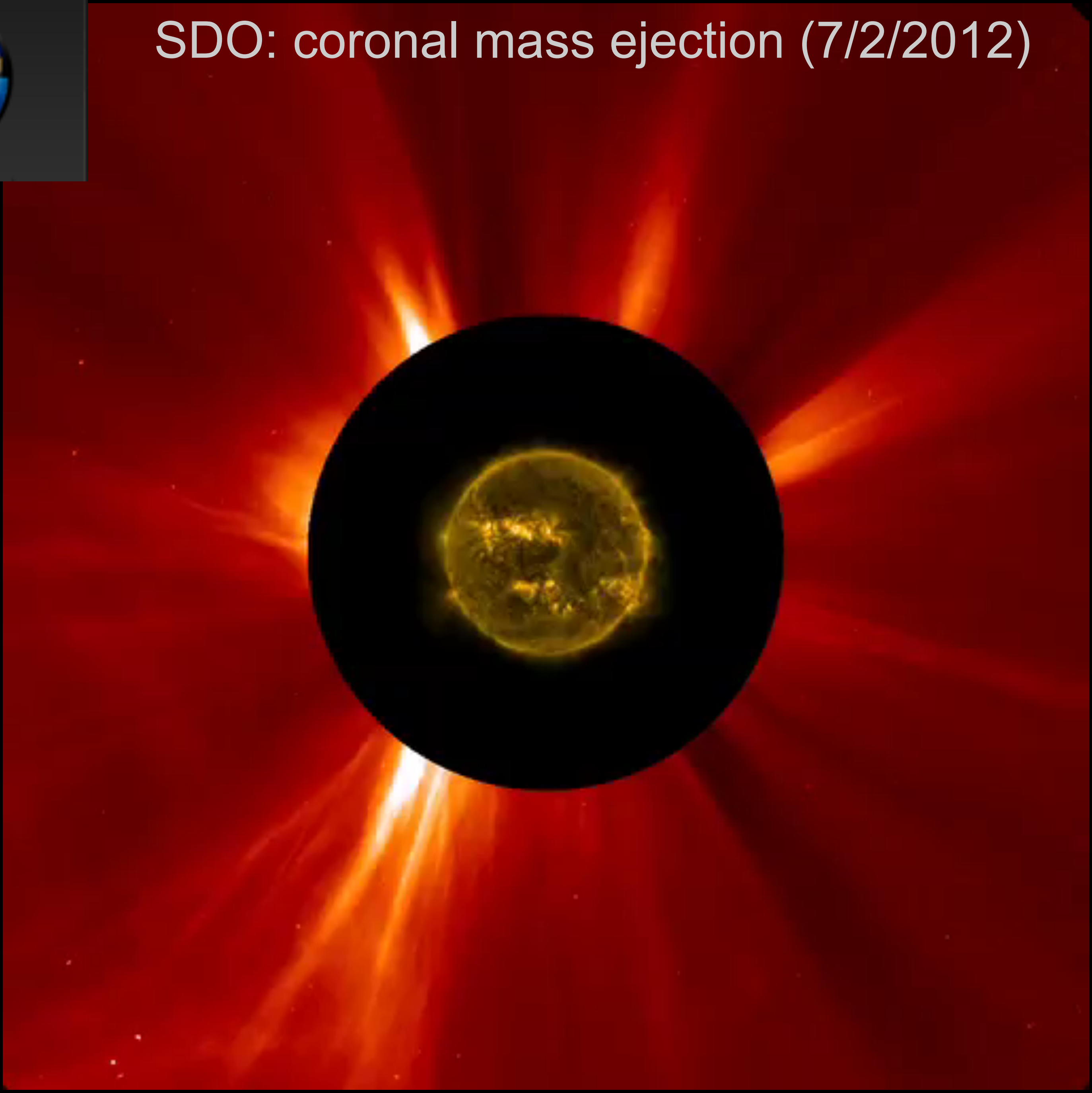
SDO: dark plasma (3/27/2012)



SDO: solar flare (4/16/2012)



SDO: coronal mass ejection (7/2/2012)



Aims of the course

- Introduce the equations needed to model the internal structure of stars.

- Overview of how basic stellar properties are observationally measured.

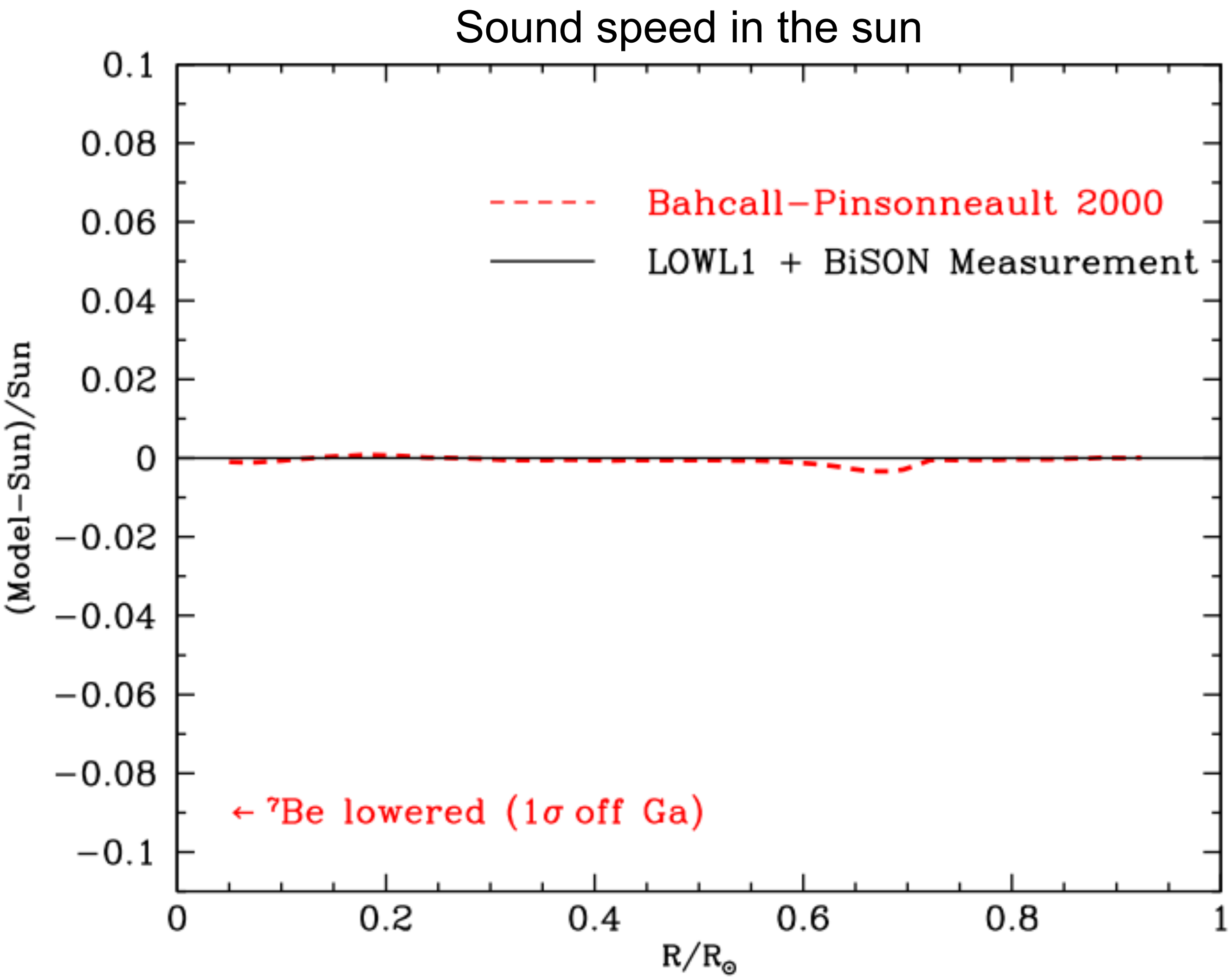
- Study the microphysics relevant for stars: the equation of state, the opacity, nuclear reactions.

- Examine the properties of simple models for stars and consider how real models are computed.

- Survey (mostly qualitatively) how stars evolve, and the endpoints of stellar evolution.

- Discuss a handful of ongoing research areas in stellar physics.

Stars are relatively simple physical systems

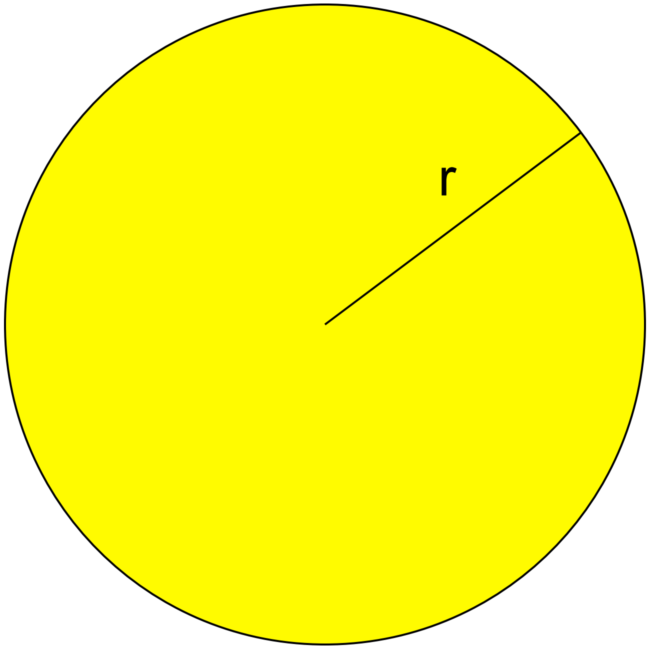


Problem of Stellar Structure

We want to determine the structure (density, temperature, energy output, pressure as a function of radius) of an isolated mass M of gas with a given composition (e.g., H, He, etc.)

Known:

Mass
+
Composition



Unknown:

Density
Temperature
Energy
Pressure

Simplifying assumptions

1. No rotation → spherical symmetry ✔

   For sun: rotation period at surface ~ 1 month
   orbital period at surface ~ few hours

2. No magnetic fields ✔

   For sun: magnetic field ~ 5G, ~ 1KG in sunspots
   equipartition field ~ 100 MG

   Some neutron stars have a large fraction of their energy in B fields

3. Static ✔

   For sun: convection, but no large scale variability

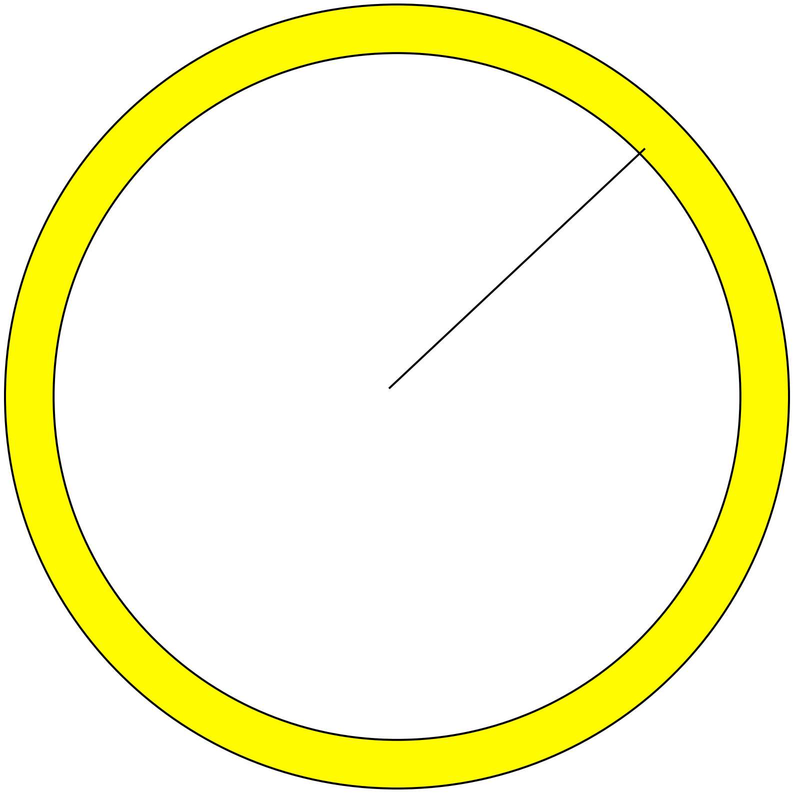
   Not valid for forming stars, pulsating stars and dying stars.

4. Newtonian gravity ✔

   For sun: escape velocity ~ 600 km/s << c

   Not true for neutron stars

The Variables of Stellar Structure



r	radius
$m(r)$	enclosed mass
$\rho(r)$	mass density
$P(r)$	Pressure
$g(r)$	gravity
$T(r)$	Temperature
$L_r(r)$	Luminosity flow
$X_i(r)$	Composition

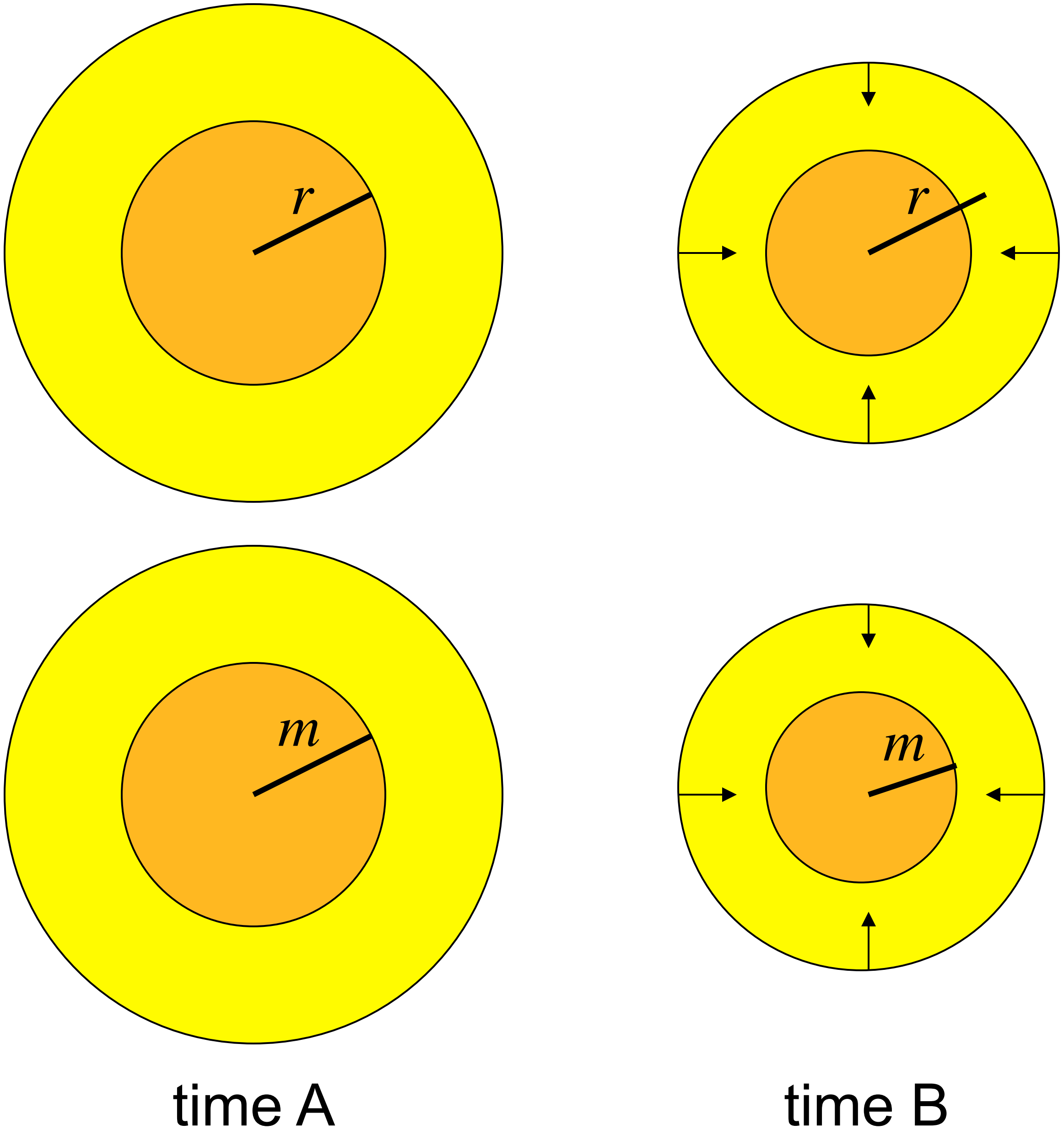
Eulerian vs. Lagrangian

Eulerian

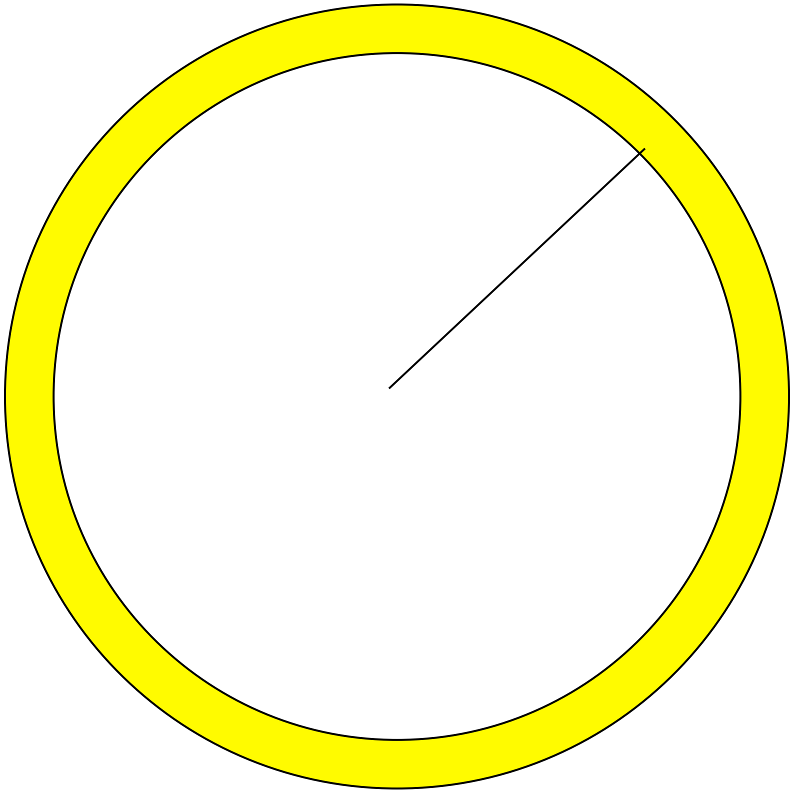
Characterize quantities as a function of radius.

Lagrangian

Characterize quantities as a function of enclosed mass.



The Variables of Stellar Structure



m	enclosed mass
$r(m)$	radius
$\rho(m)$	mass density
$P(m)$	Pressure
$g(m)$	gravity
$T(m)$	Temperature
$L_r(m)$	Luminosity flow
$X_i(m)$	Composition

Values for the Sun

$$\text{Mass} \equiv 1M_\odot \approx 2\times10^{33}\,\text{g}$$

$$\text{Radius} \equiv 1R_\odot \approx 7\times10^{10}\,\text{cm}$$

$$\text{Luminosity} \equiv 1L_\odot \approx 4\times10^{33}\,\text{erg/s}$$

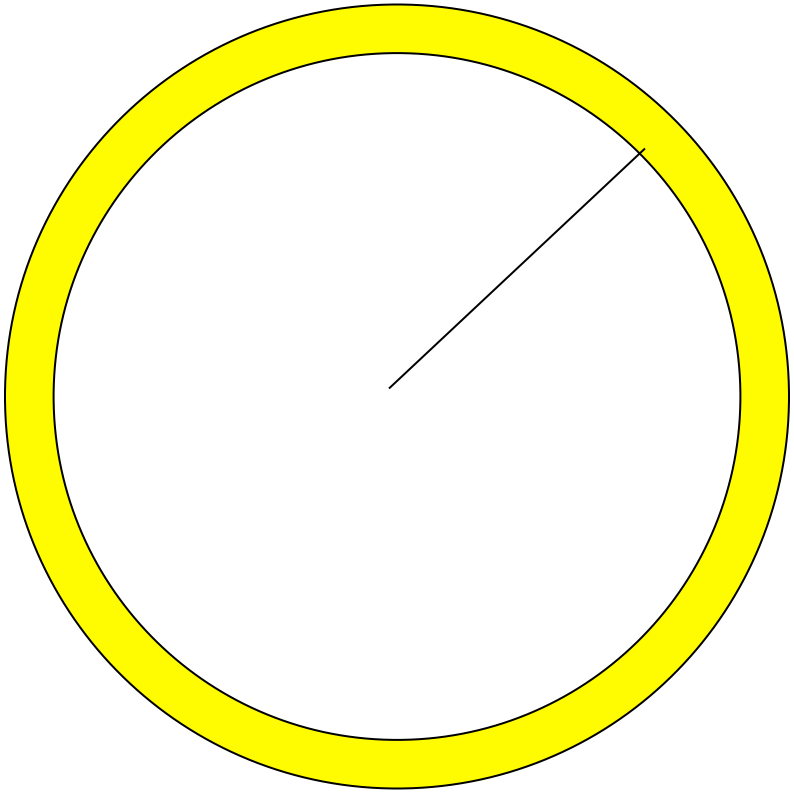
$$\text{Surface Temperature} \approx 5{,}800\ \text{K}$$

$$\text{Composition} \approx 70\%\ \text{H} \quad 28\%\ \text{He} \quad 2\%\ \text{metals}$$

The Equations of Stellar Structure

mass in shell = density × volume of shell

$$dm = \rho \times 4\pi r^2 dr$$



The Equations of Stellar Structure

1. Mass Conservation

$$\boxed{\frac{dm}{dr} = 4\pi r^2 \rho}$$

eulerian

$$\boxed{\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}}$$

lagrangian

The Equations of Stellar Structure

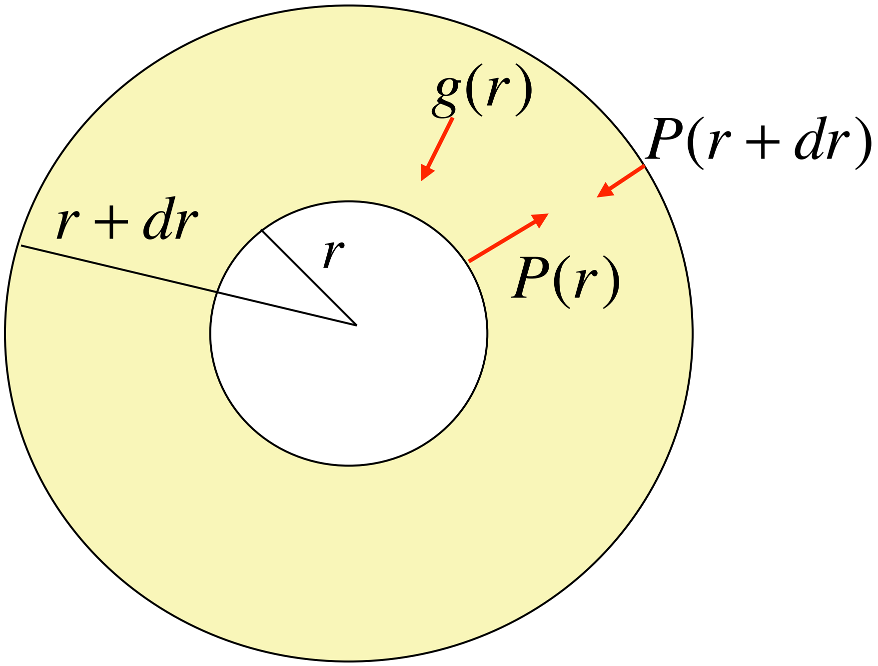
$$F = ma$$

Force pushing outwards:

$$P(r)4\pi r^2$$

Force pushing inwards:

$$-P(r+dr)4\pi r^2 - \frac{Gm}{r^2}dm$$



$$P(r)4\pi r^2 - P(r+dr)4\pi r^2 - \frac{Gm}{r^2}dm = \ddot{r}dm$$

$$\rightarrow -dP4\pi r^2 - \frac{Gm}{r^2}dm = \ddot{r}dm \qquad \rightarrow -\frac{dP}{dm}4\pi r^2 - \frac{Gm}{r^2} = \ddot{r} \qquad = 0$$

The Equations of Stellar Structure

2. Hydrostatic Equilibrium

$$\boxed{\frac{dP}{dr} = -\rho \frac{Gm}{r^2}}$$

eulerian

$$\boxed{\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}}$$

lagrangian

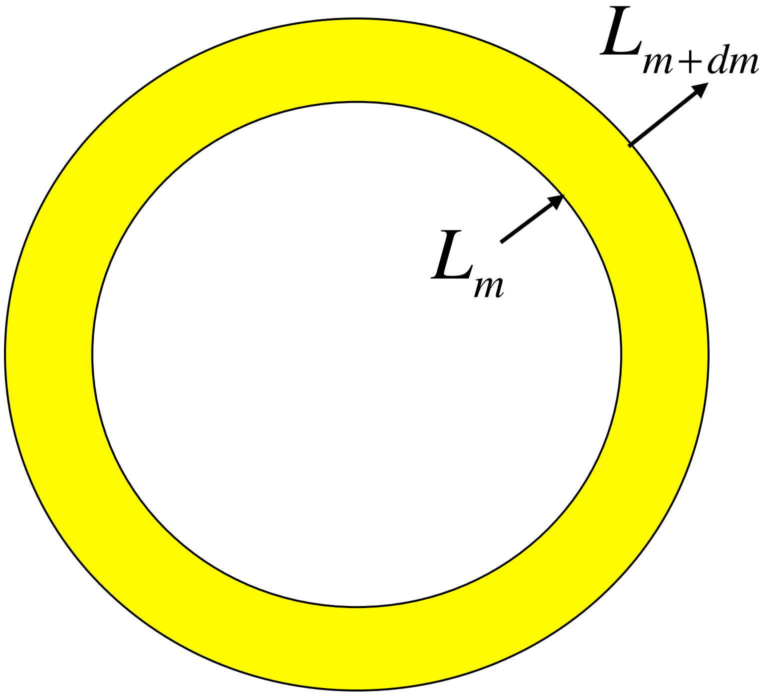
Need equation of state: $P = f\left(\rho, T, X_i\right)$

e.g., for an ideal gas: $P = \frac{R}{\mu}\rho T$

The Equations of Stellar Structure

ε = energy generation rate per unit mass (erg/s/g)

change in $L_m = \varepsilon \times dm$



The Equations of Stellar Structure

3. Energy Generation

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon}$$

$$\boxed{\frac{dL_m}{dm} = \varepsilon}$$

eulerian

lagrangian

Need nuclear physics: $\varepsilon = f\left(\rho, T, X_i\right)$

e.g., for the proton-proton chain: $\varepsilon \approx \varepsilon_0 \rho T^4$

The Equations of Stellar Structure

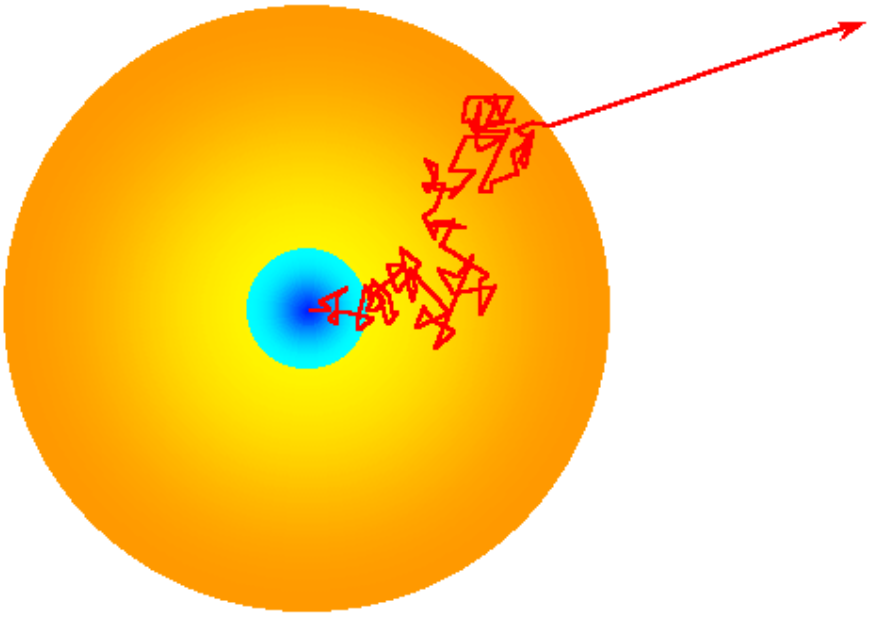
Energy flow: Depends on opacity κ (area/mass)

- radiation *low opacity*
- convection *high opacity*
- conduction *unimportant*

For radiation:

radiation pressure decreases outward ➔
photons have net movement outward in their random walk.

Random walk of photon through the sun.



Straight path: 2.3 seconds
Random walk: 30,000 years

The Equations of Stellar Structure

4. Energy Flow

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acr^2T^3}$$

$$\frac{dT}{dm} = -\frac{3\kappa L_m}{64\pi^2 acr^4T^3}$$

eulerian

lagrangian

Need opacity: $\kappa = f\left(\rho, T, X_i\right)$

e.g., for electron Thomson scattering: $\kappa \approx \kappa_0$

Solving the Equations

Need microphysics: $P(\rho, T, X_i)$, $\varepsilon(\rho, T, X_i)$, $\kappa(\rho, T, X_i)$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dL_m}{dm} = \varepsilon$$

$$\frac{dT}{dm} = -\frac{3\kappa L_m}{64\pi^2 acr^4 T^3}$$

Solve in 1 dimension from center to surface

Boundary Conditions

$$\text{Center } (r=0): \quad m=0, \quad L_r=0, \quad P=P_c, \quad T=T_c$$

$$\text{Surface } (r=R): \quad m=M, \quad L_r=L, \quad P\sim 0, \quad T\sim 0$$

Lagrangian

$$\text{Center } (m=0): \quad r=0, \quad L_r=0, \quad P=?, \quad T=?$$

$$\text{Surface } (m=M): \quad r=?, \quad L_r=?, \quad P\sim 0, \quad T\sim 0$$

Boundary conditions are incomplete at each end

Complications

1. Stars are luminous

   radiate away energy → must change in time

   chemical composition is changing. Must account for dX_i/dt

2. Convection is very complicated and important

3. Convection can change chemical composition

4. Opacities are hard to calculate. And they matter!

Homology Relations

To calculate the luminosity or radius of a star of mass *M*, we must solve these differential equations. However, we can get approximate scaling relations by using *Homology.*

Assume that each differential or local quantity simply scales with the global value of that quantity.

For example,

$$\frac{dr}{dm} \sim \frac{R}{M}$$

So,

$$\boxed{\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}}$$

becomes

$$\boxed{\frac{R}{M} \sim \frac{1}{R^2 \overline{\rho}}}$$

Homology Relations

If we also know how pressure, opacity, and nuclear generation rate scale with density and temperature, we can solve all these equations to get scaling relations. Then we can turn these scaling relations into actual equations by normalizing to the Sun.

For example, suppose we find that $L \sim M^{\alpha}$

This means that $L = \mathrm{const} \times M^{\alpha}$

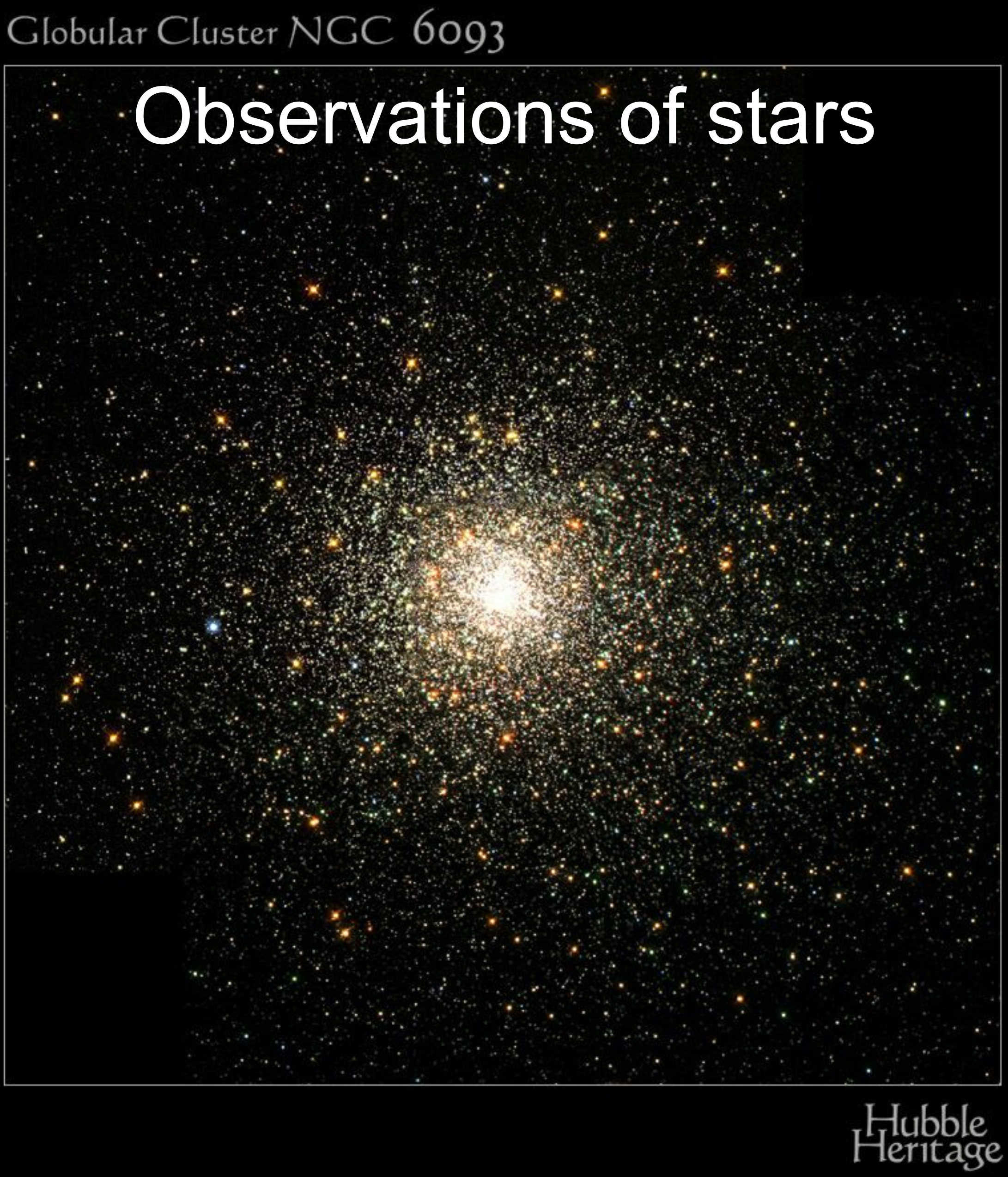
For the Sun, this is $L_{\odot} = \mathrm{const} \times M_{\odot}^{\alpha}$

Dividing the two equations we get $$\boxed{\frac{L}{L_{\odot}} = \left(\frac{M}{M_{\odot}} \right)^{\alpha}}$$

Observations of stars

Hyades open cluster

Observations of stars



PRC99-26 • Space Telescope Science Institute • Hubble Heritage Team (AURA/STScI/NASA)

Astronomers count photons!

We can measure three things about a photon:

- What direction did it come from?
- When did it arrive?
- What was its energy (wavelength/color)?

Different types of observing modes

1. Count photons in a fixed region of the sky (aperture), a fixed window of time (exposure time), and a fixed wavelength range (filter or band).

   what we call "*photometry*"

2. Count photons <u>as a function of direction</u> in a fixed window of time (exposure time), and a fixed wavelength range (filter or band).

   what we call "*imaging*"

Different types of observing modes

3. Count photons as a function of wavelength in a fixed region of the sky (aperture), and a fixed window of time (exposure time).

   what we call "*spectroscopy*"

4. Count photons as a function of time in a fixed region of the sky (aperture), and a fixed wavelength range (filter or band).

   what we call "*time series photometry*"

Different types of observing modes

5. Count photons as a function of direction AND time in a fixed wavelength range (filter or band).

   what we call "*time series imaging*"

6. Count photons as a function of wavelength AND time in a fixed region of the sky (aperture).

   what we call "*time series spectroscopy*"

Different types of observing modes

7. Count photons as a function of direction AND wavelength in a fixed window of time (exposure time).

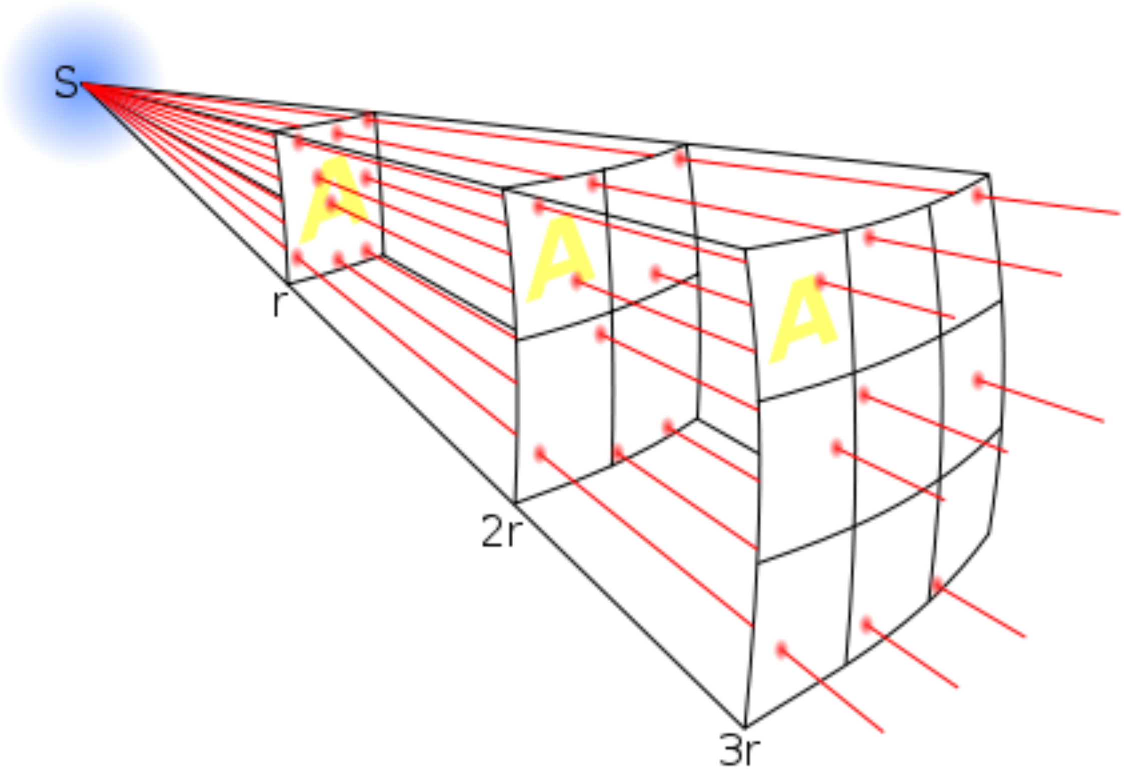
   what we call “*integral field spectroscopy*”

8. Count photons as a function of direction AND time AND wavelength.

   the holy grail of observational astronomy…

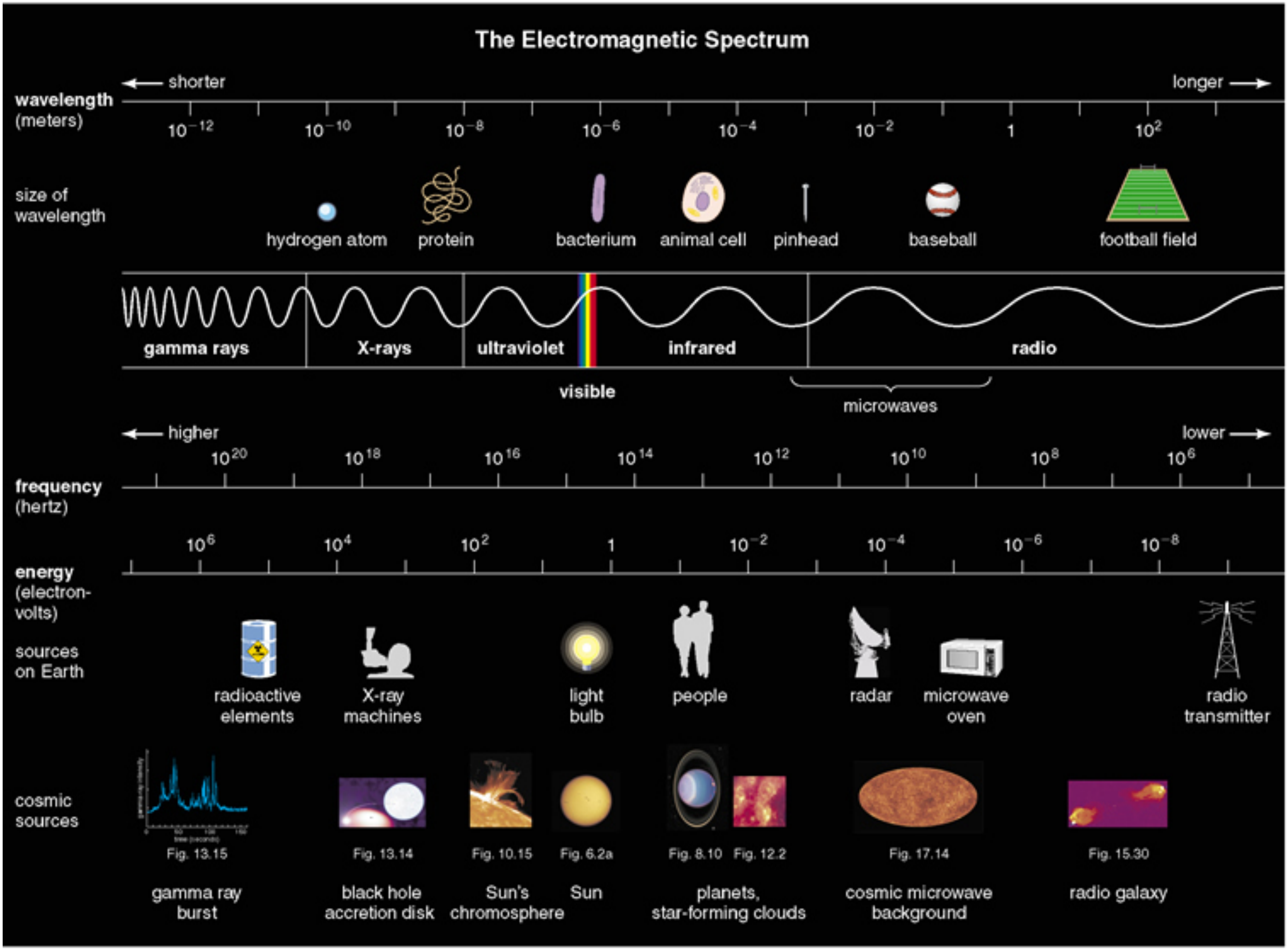
Luminosity and flux

Luminosity	L : energy/time (erg/s)
Flux	f : luminosity/area (erg/s/cm^2)

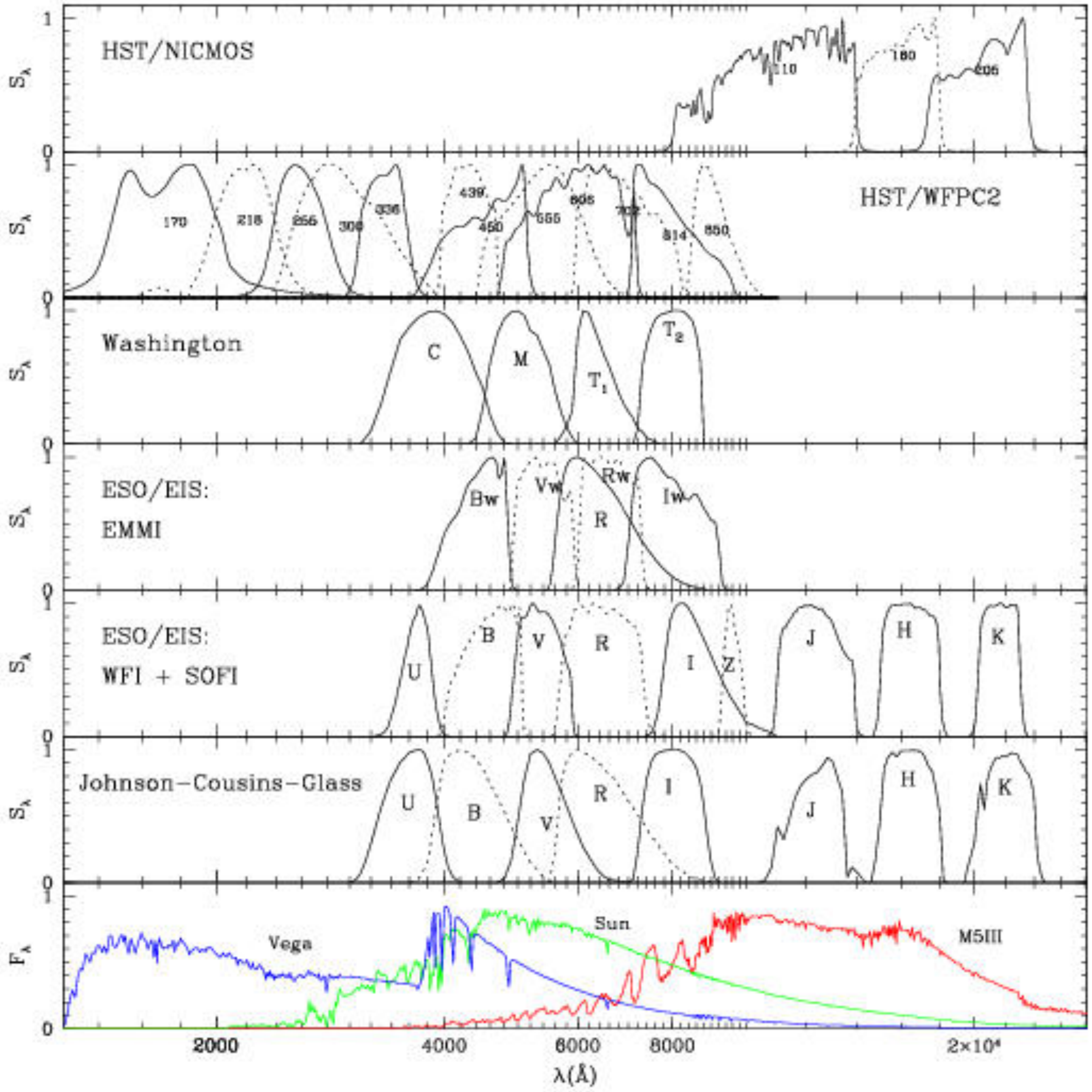


Inverse square law:

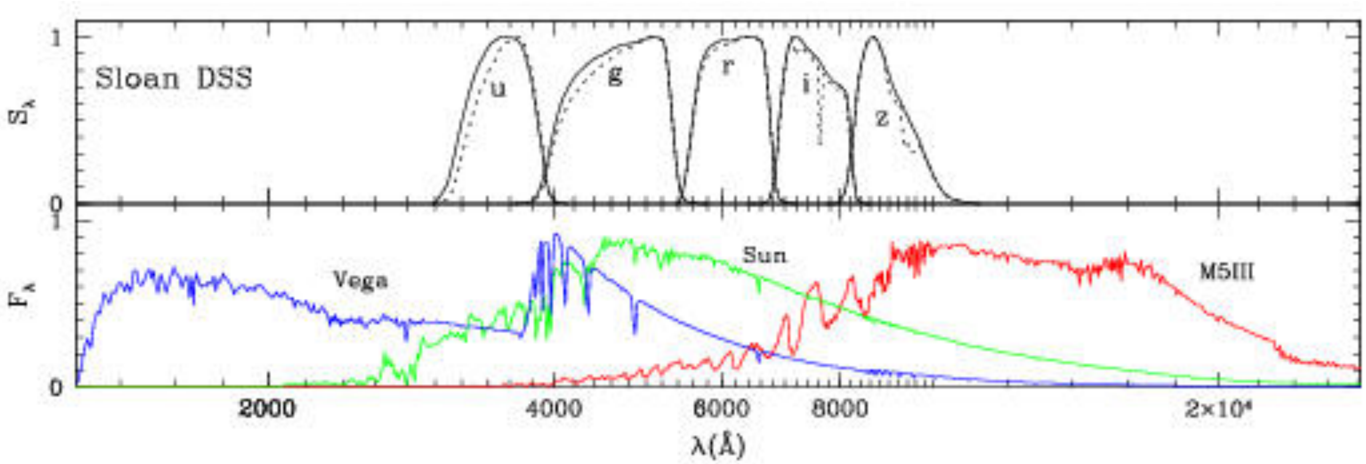
$$\boxed{f = \frac{L}{4\pi d^2}}$$


The Electromagnetic Spectrum
shorter
longer
wavelength (meters)
10^{-12}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
1
10^{2}
size of wavelength
hydrogen atom
protein
bacterium
animal cell
pinhead
baseball
football field
gamma rays
X-rays
ultraviolet
infrared
radio
visible
microwaves
higher
lower
10^{20}
10^{18}
10^{16}
10^{14}
10^{12}
10^{10}
10^{8}
10^{6}
frequency (hertz)
10^{6}
10^{4}
10^{2}
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
energy (electron-volts)
sources on Earth
radioactive elements
X-ray machines
light bulb
people
radar
microwave oven
radio transmitter
cosmic sources
Fig. 13.15
gamma ray burst
Fig. 13.14
black hole accretion disk
Fig. 10.15
Sun's chromosphere
Fig. 6.2a
Sun
Fig. 8.10
Fig. 12.2
planets, star-forming clouds
Fig. 17.14
cosmic microwave background
Fig. 15.30
radio galaxy

Photometric filters



SDSS filters



Photometric Filters

Astronomical fluxes are usually measured using filters

The flux in the SDSS *g* filter is:

$$f_g = \int_0^\infty F(\lambda) S_g(\lambda) d\lambda$$

The flux in the SDSS *r* filter is:

$$f_r = \int_0^\infty F(\lambda) S_r(\lambda) d\lambda$$

Apparent magnitude

$$\boxed{m = -2.5 \log f + const}$$

A star that is 5 magnitudes brighter (smaller m) has 100x the flux.

$$\boxed{\begin{aligned} & m_1 - m_2 = -2.5 \log\left(f_1 / f_2\right) \\ & \frac{f_1}{f_2} = 10^{(m_2 - m_1)/2.5} \end{aligned}}$$

Absolute magnitude

M = apparent magnitude the star would have if it were 10pc away.

$$f = \frac{L}{4\pi d^2} \qquad f_{10} = \frac{L}{4\pi\left(10\,pc\right)^2}$$

$$m - M = -2.5\log\left[\frac{L/4\pi d^2}{L/4\pi\left(10\,pc\right)^2}\right] = -2.5\log\left[\left(\frac{d}{10\,pc}\right)^{-2}\right]$$

$$m - M = \underbrace{5\log\left(\frac{d}{10\,pc}\right)}_{\text{distance modulus}}$$

Hipparcos Data

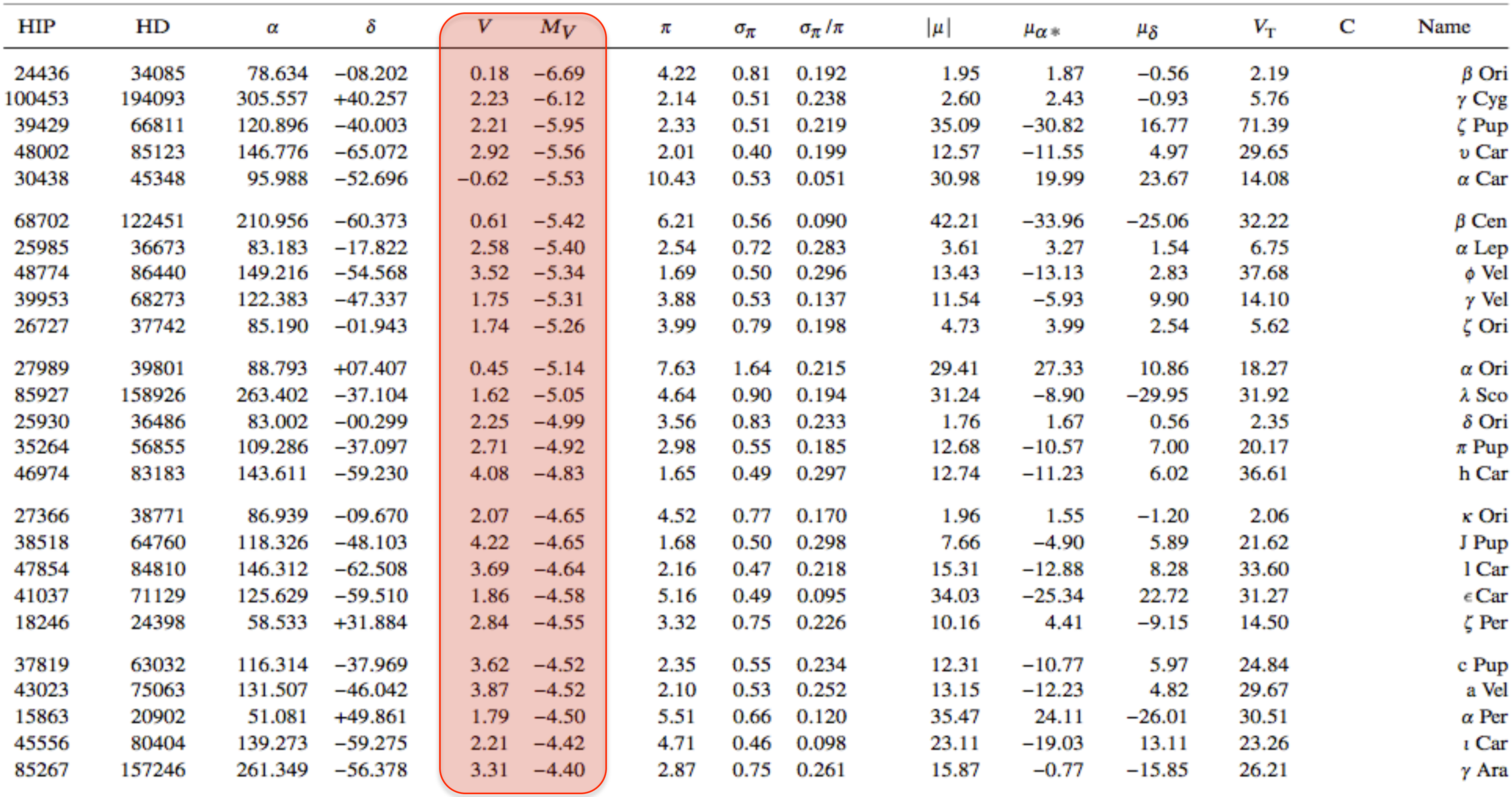


Table 3.6.4. The 150 most luminous stars in the Hipparcos Catalogue.

HIP	HD	α	δ	V	M_V	π	σ_π	σ_π/π	$\lvert\mu\rvert$	$\mu_{\alpha*}$	μ_δ	V_T	C	Name
24436	34085	78.634	−08.202	0.18	−6.69	4.22	0.81	0.192	1.95	1.87	−0.56	2.19		β Ori
100453	194093	305.557	+40.257	2.23	−6.12	2.14	0.51	0.238	2.60	2.43	−0.93	5.76		γ Cyg
39429	66811	120.896	−40.003	2.21	−5.95	2.33	0.51	0.219	35.09	−30.82	16.77	71.39		ζ Pup
48002	85123	146.776	−65.072	2.92	−5.56	2.01	0.40	0.199	12.57	−11.55	4.97	29.65		υ Car
30438	45348	95.988	−52.696	−0.62	−5.53	10.43	0.53	0.051	30.98	19.99	23.67	14.08		α Car
68702	122451	210.956	−60.373	0.61	−5.42	6.21	0.56	0.090	42.21	−33.96	−25.06	32.22		β Cen
25985	36673	83.183	−17.822	2.58	−5.40	2.54	0.72	0.283	3.61	3.27	1.54	6.75		α Lep
48774	86440	149.216	−54.568	3.52	−5.34	1.69	0.50	0.296	13.43	−13.13	2.83	37.68		φ Vel
39953	68273	122.383	−47.337	1.75	−5.31	3.88	0.53	0.137	11.54	−5.93	9.90	14.10		γ Vel
26727	37742	85.190	−01.943	1.74	−5.26	3.99	0.79	0.198	4.73	3.99	2.54	5.62		ζ Ori
27989	39801	88.793	+07.407	0.45	−5.14	7.63	1.64	0.215	29.41	27.33	10.86	18.27		α Ori
85927	158926	263.402	−37.104	1.62	−5.05	4.64	0.90	0.194	31.24	−8.90	−29.95	31.92		λ Sco
25930	36486	83.002	−00.299	2.25	−4.99	3.56	0.83	0.233	1.76	1.67	0.56	2.35		δ Ori
35264	56855	109.286	−37.097	2.71	−4.92	2.98	0.55	0.185	12.68	−10.57	7.00	20.17		π Pup
46974	83183	143.611	−59.230	4.08	−4.83	1.65	0.49	0.297	12.74	−11.23	6.02	36.61		h Car
27366	38771	86.939	−09.670	2.07	−4.65	4.52	0.77	0.170	1.96	1.55	−1.20	2.06		κ Ori
38518	64760	118.326	−48.103	4.22	−4.65	1.68	0.50	0.298	7.66	−4.90	5.89	21.62		J Pup
47854	84810	146.312	−62.508	3.69	−4.64	2.16	0.47	0.218	15.31	−12.88	8.28	33.60		l Car
41037	71129	125.629	−59.510	1.86	−4.58	5.16	0.49	0.095	34.03	−25.34	22.72	31.27		ε Car
18246	24398	58.533	+31.884	2.84	−4.55	3.32	0.75	0.226	10.16	4.41	−9.15	14.50		ζ Per
37819	63032	116.314	−37.969	3.62	−4.52	2.35	0.55	0.234	12.31	−10.77	5.97	24.84		c Pup
43023	75063	131.507	−46.042	3.87	−4.52	2.10	0.53	0.252	13.15	−12.23	4.82	29.67		a Vel
15863	20902	51.081	+49.861	1.79	−4.50	5.51	0.66	0.120	35.47	24.11	−26.01	30.51		α Per
45556	80404	139.273	−59.275	2.21	−4.42	4.71	0.46	0.098	23.11	−19.03	13.11	23.26		ι Car
85267	157246	261.349	−56.378	3.31	−4.40	2.87	0.75	0.261	15.87	−0.77	−15.85	26.21		γ Ara

Bolometric magnitudes

M is measured in a band. To get the light from all wavelengths, we must add a correction.

Bolometric correction:

$$M_{bol} = M_V + BC$$

$$m_{bol} = m_V + BC$$

BC depends on band and star spectrum. By definition, BC=0 for V-band and T=6600K

Color

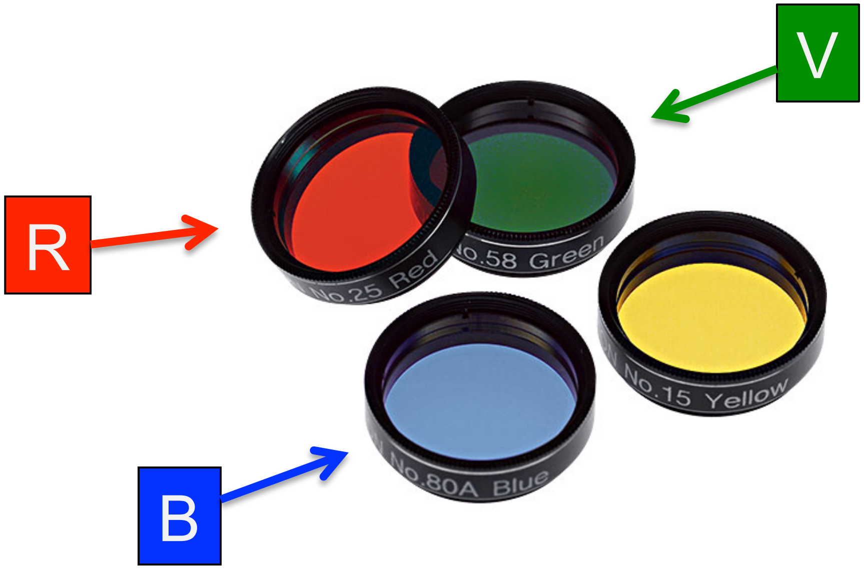
Color = crude, low resolution, estimate of spectral shape

$$B - V = m_B - m_V = M_B - M_V = -2.5\log\left(\frac{f_B}{f_V}\right)$$

- distance independent
- indicator of surface temperature
- by definition, *(B-V)*=0 for Vega (T~9500K)

Color

- Measure a star's brightness through two different filters

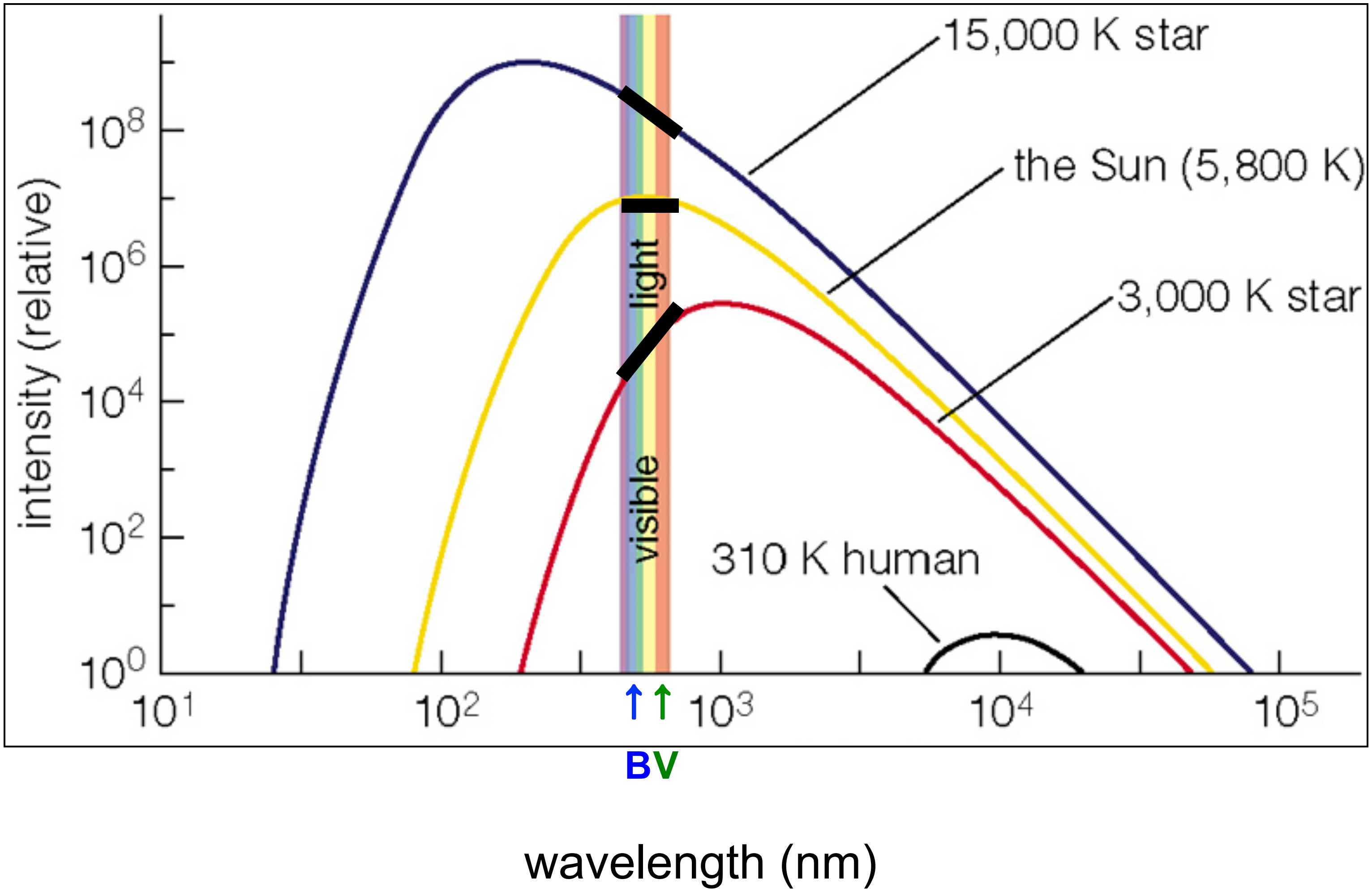


- Take the ratio of brightness: (redder filter)/(bluer filter)

  if ratio is large → red star
  if ratio is small → blue star

e.g., **V**/**B**

Color



The color of a star measured like this tells us its temperature!

Stellar spectra

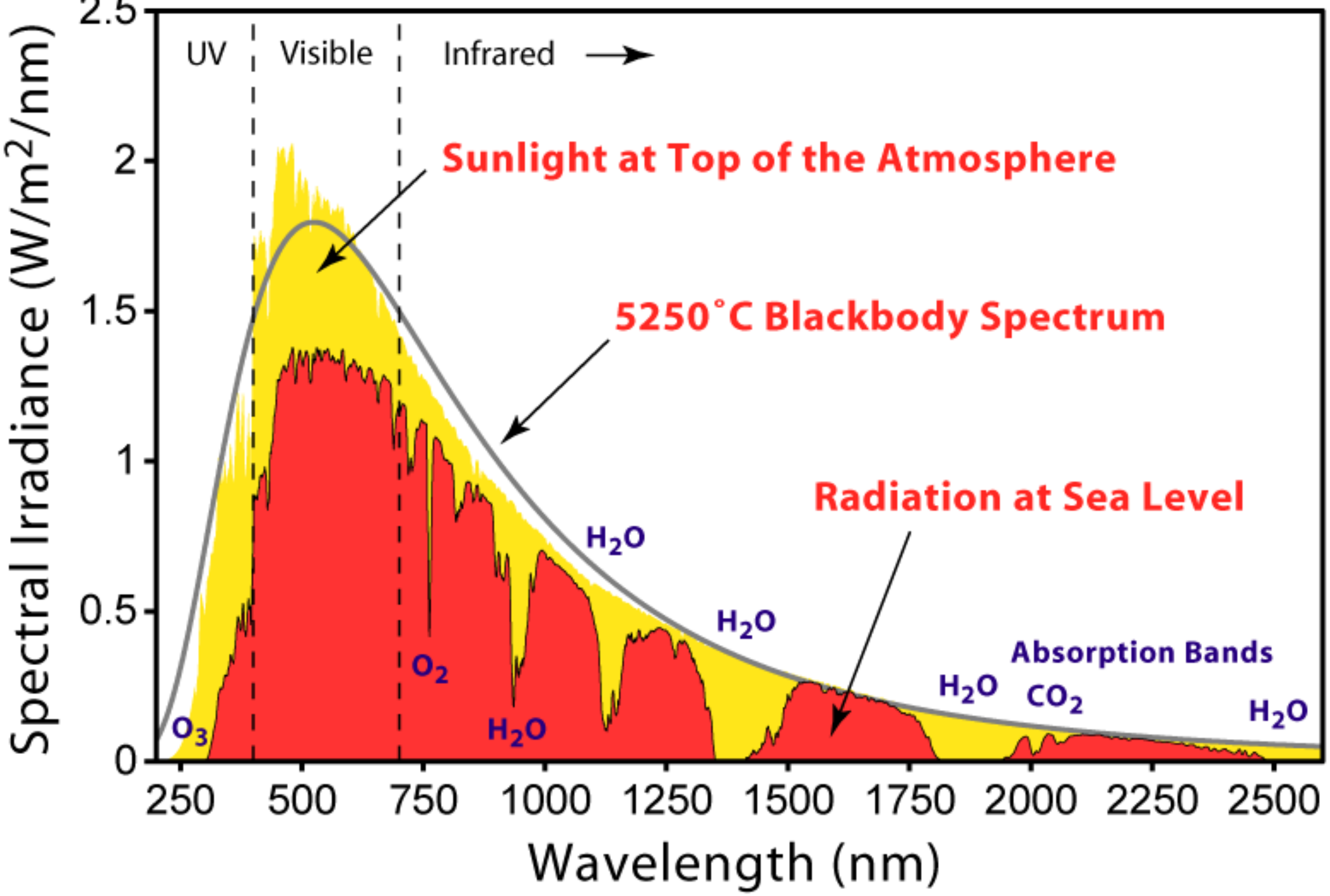
The solar spectrum can be approximated as

- a blackbody

+

- absorption lines (looking at hotter layers through cooler outer layers)

Solar Radiation Spectrum



Blackbody radiation

$$\boxed{B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}}$$

erg s^{-1} cm^{-2} Hz^{-1} st^{-1}

$$h\nu_{\max} \sim 2.8kT$$

- Effective temperature of a star = T of a blackbody that gives the same Luminosity per unit surface area of the star.

$$\boxed{L = 4\pi R^2 \sigma T_e^{\,4}}$$

Stefan–Boltzmann law

For sun: T_e = 5,778 K

Blackbody Spectrum or thermal spectrum

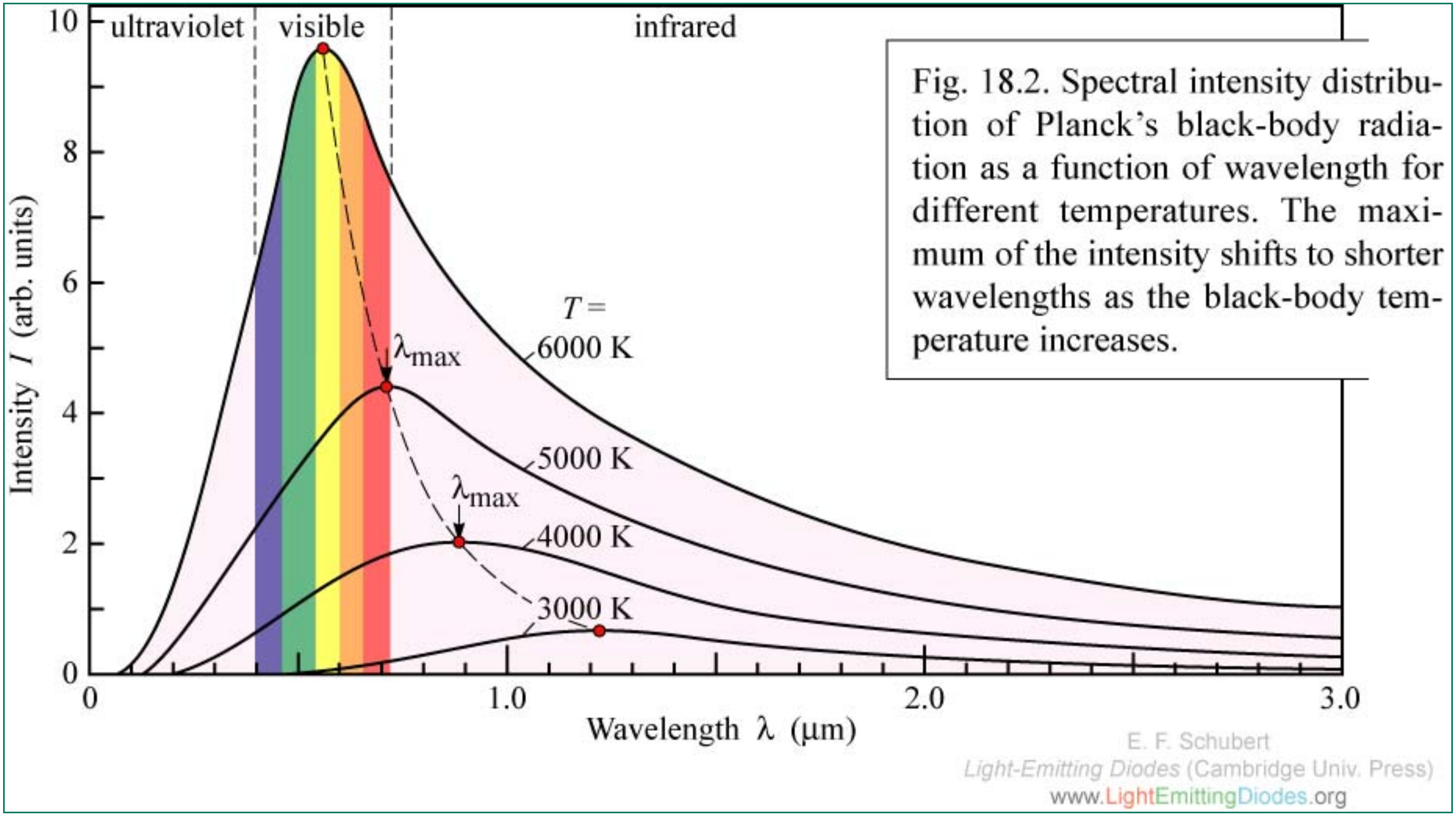
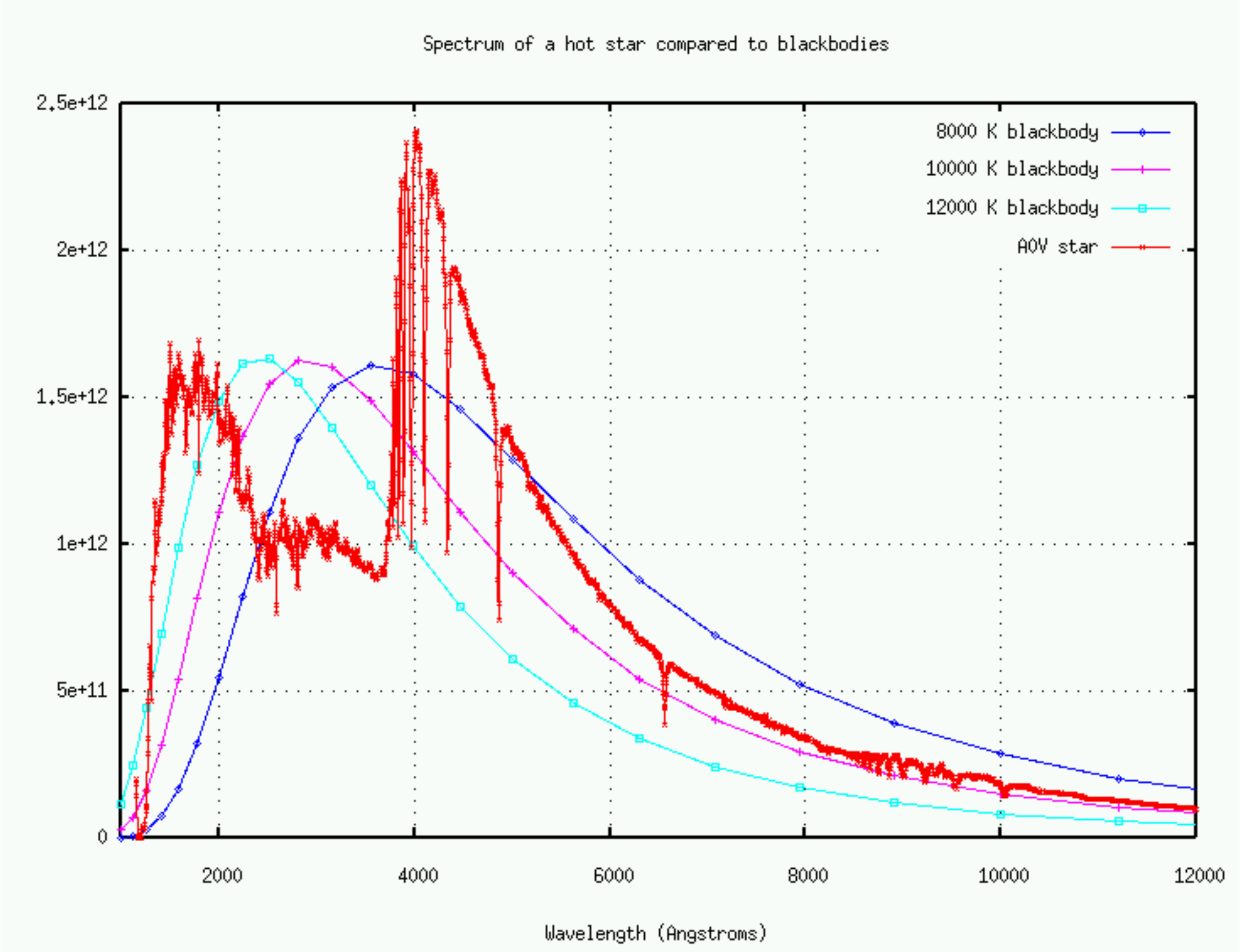


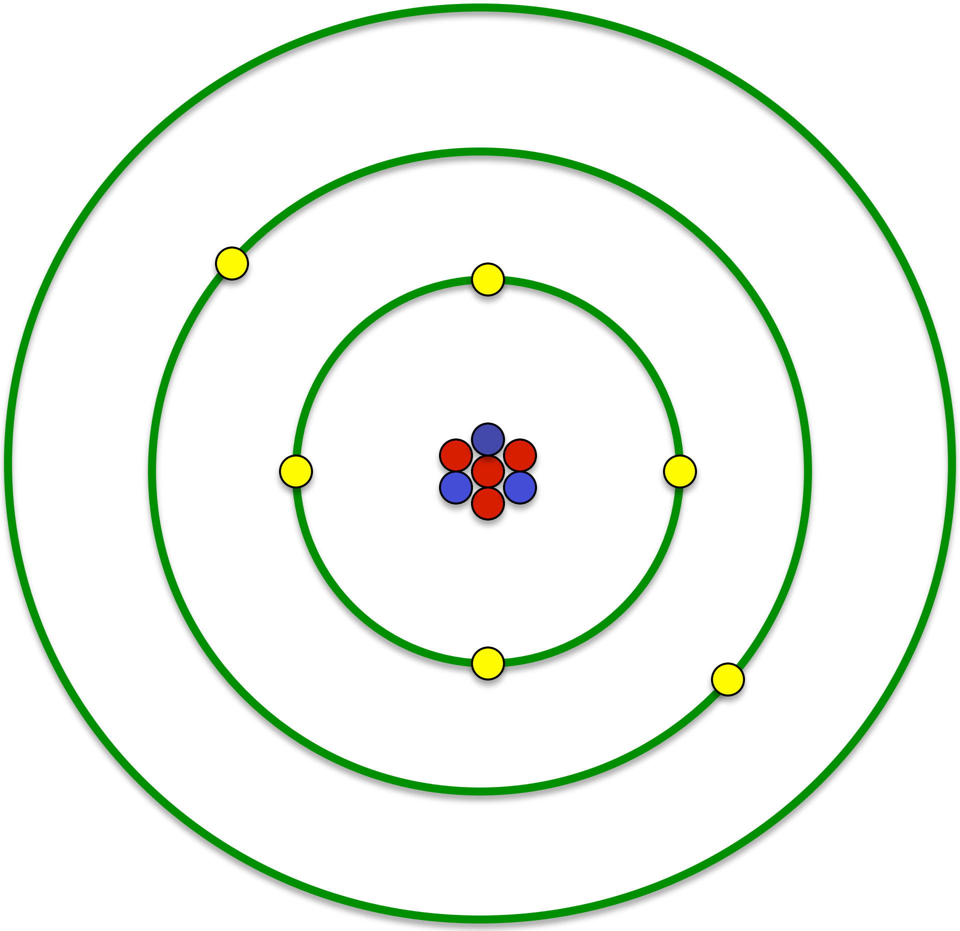
Fig. 18.2. Spectral intensity distribution of Planck's black-body radiation as a function of wavelength for different temperatures. The maximum of the intensity shifts to shorter wavelengths as the black-body temperature increases.

E. F. Schubert
Light-Emitting Diodes (Cambridge Univ. Press)
www.LightEmittingDiodes.org

Stellar spectra are not perfect blackbodies

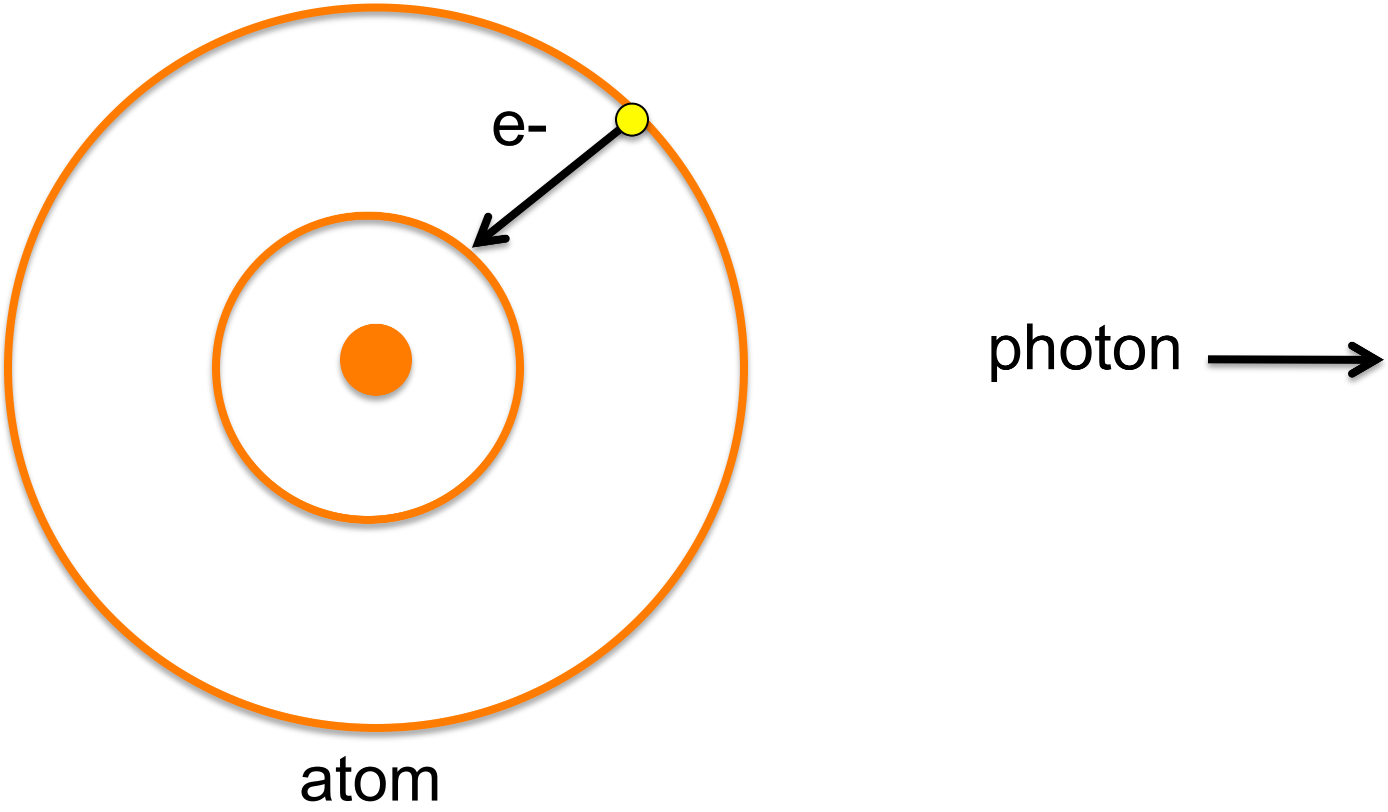


Atomic energy levels



- electrons orbit the nucleus in specific energy levels
- electrons can jump between energy levels given the right energy

Emission of light

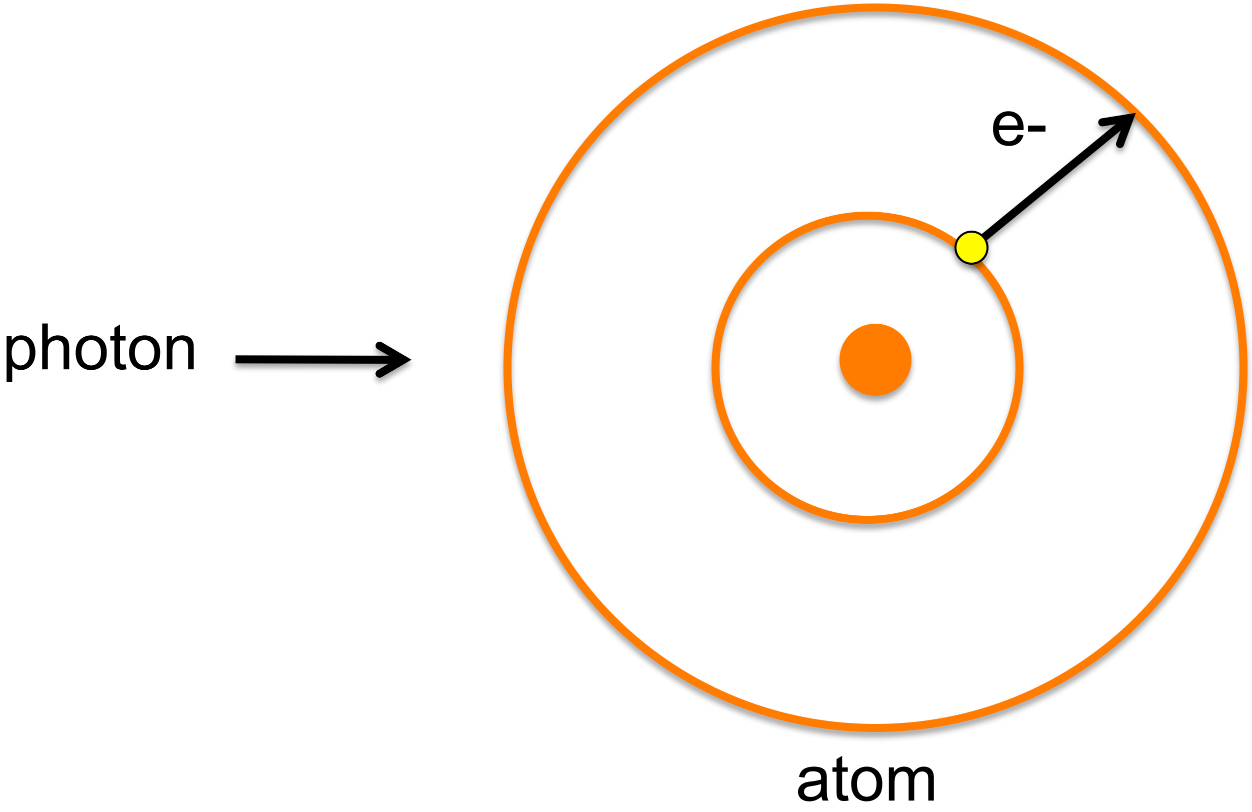


e-

photon

atom

Absorption of light

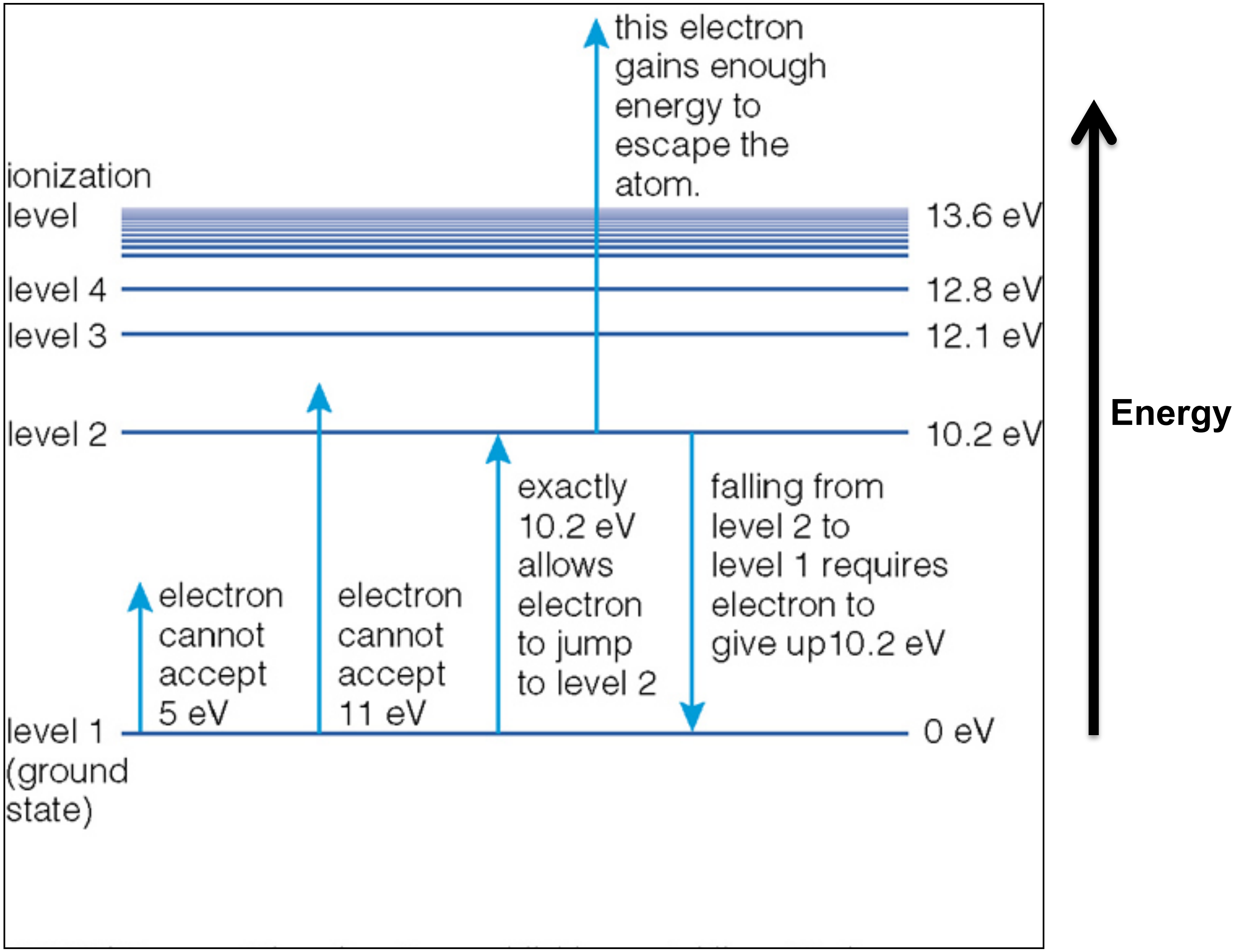


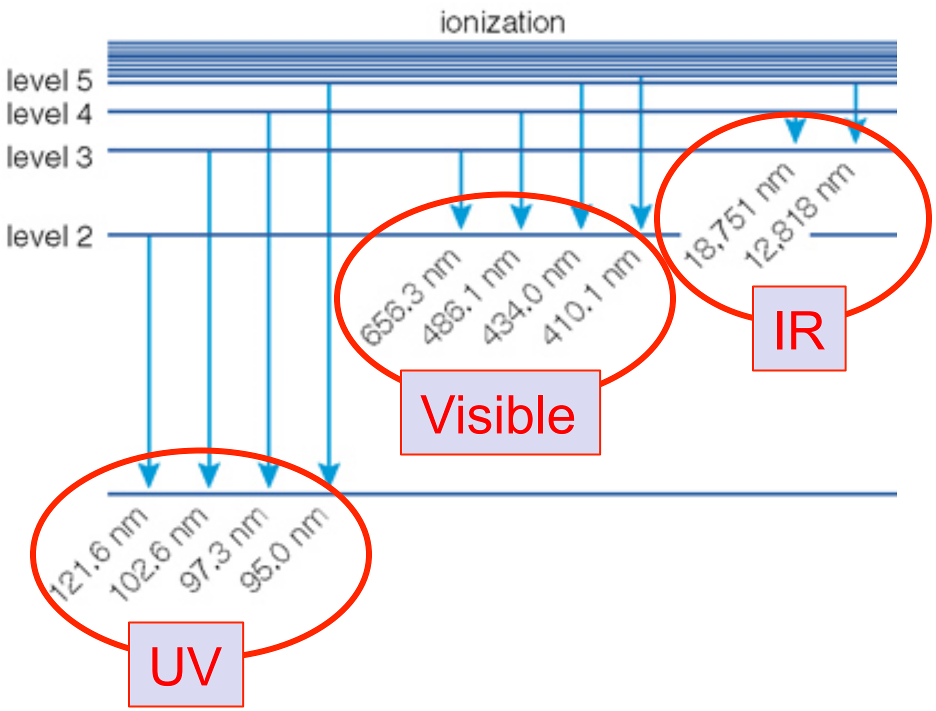
photon

e-

atom

Energy levels for Hydrogen



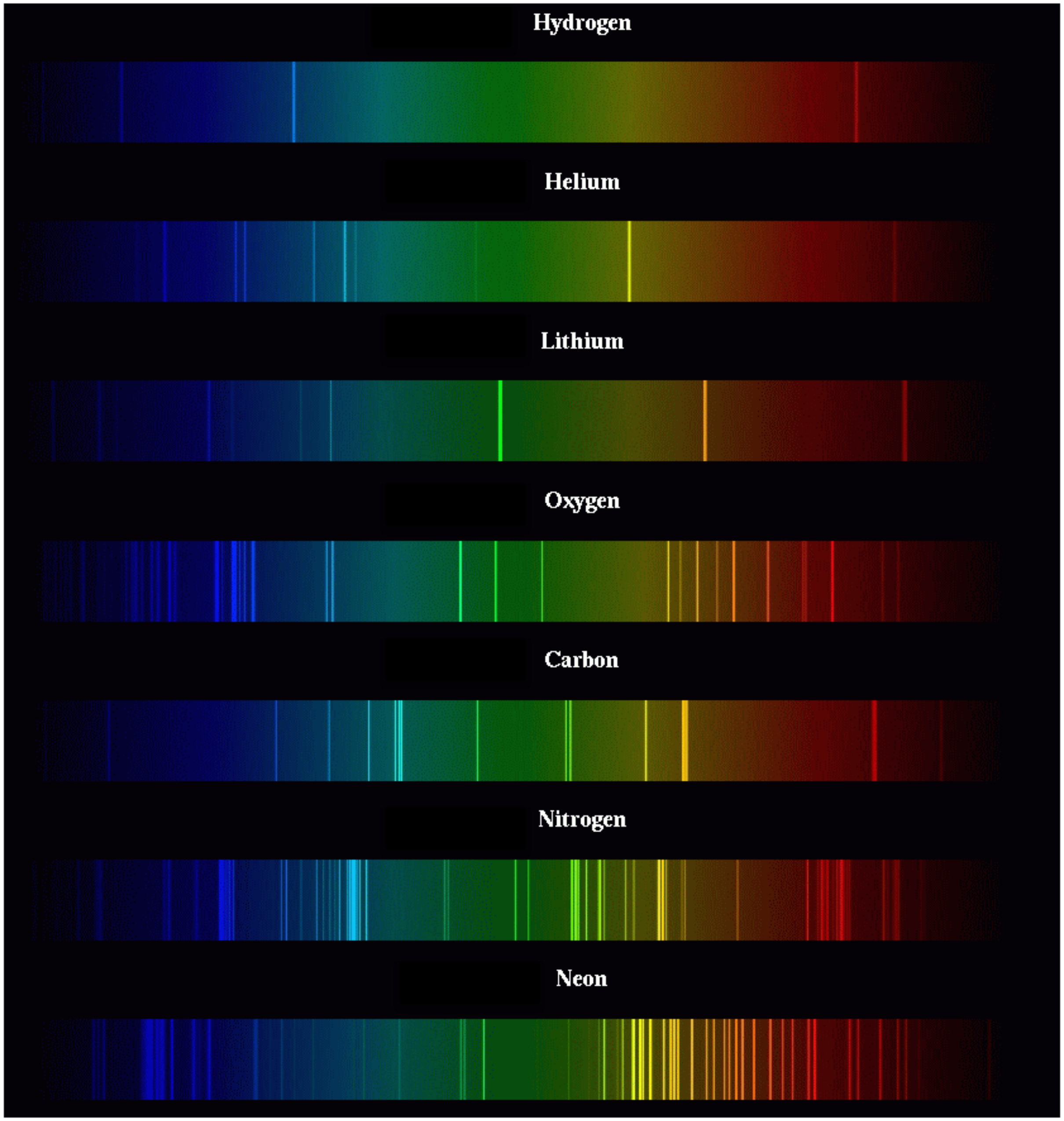


Visible spectrum shows signature of hydrogen atoms

$$E = h\nu = \frac{hc}{\lambda}$$

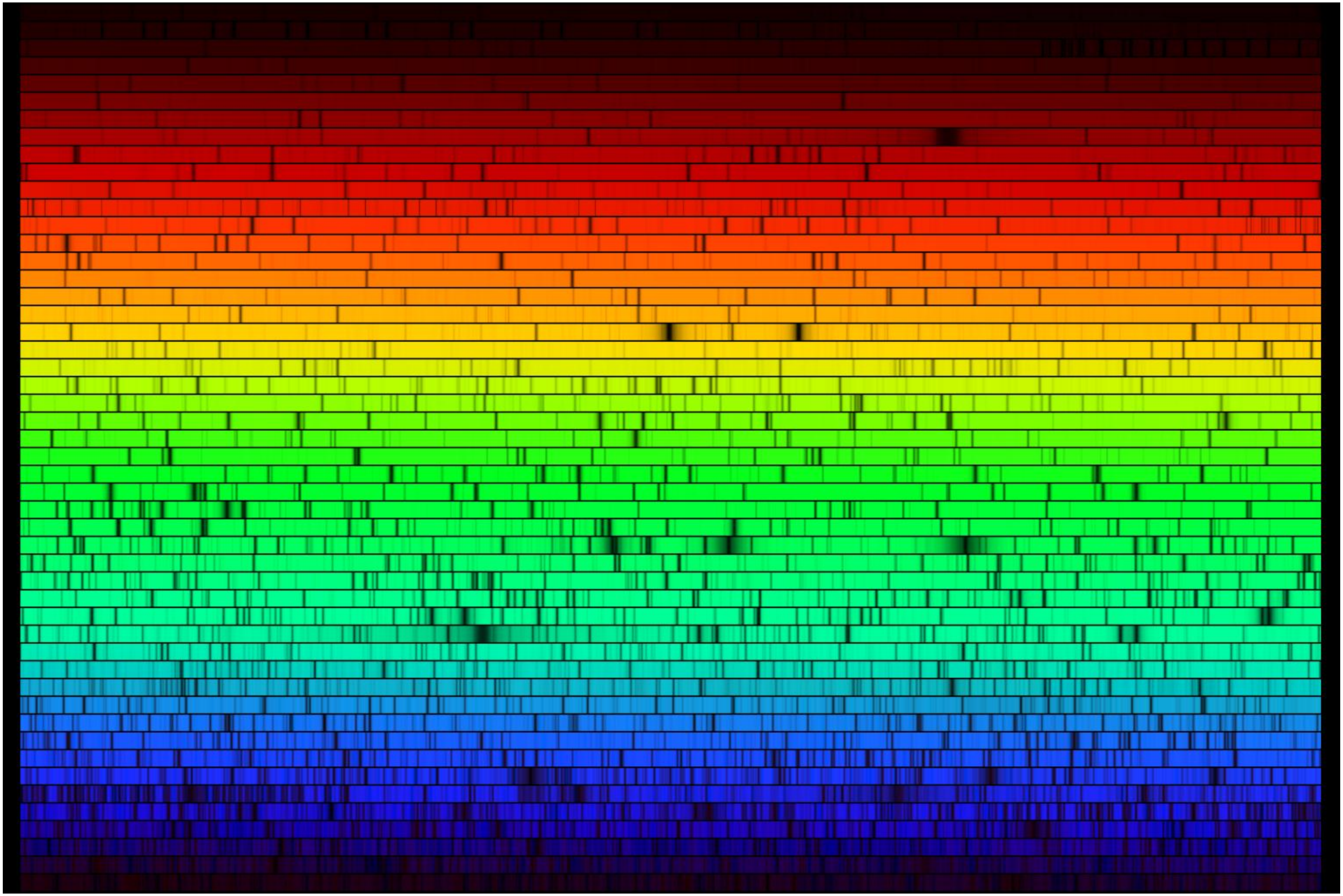
Emission line spectrum

Absorption line spectrum



Wavelength of light

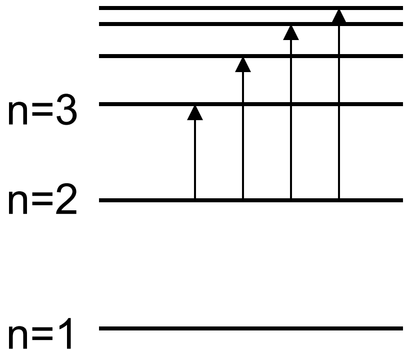
Spectrum of Sun



Spectral lines

Strength of lines depends on temperature.

e.g., Balmer lines:
transitions from n=2 to higher states



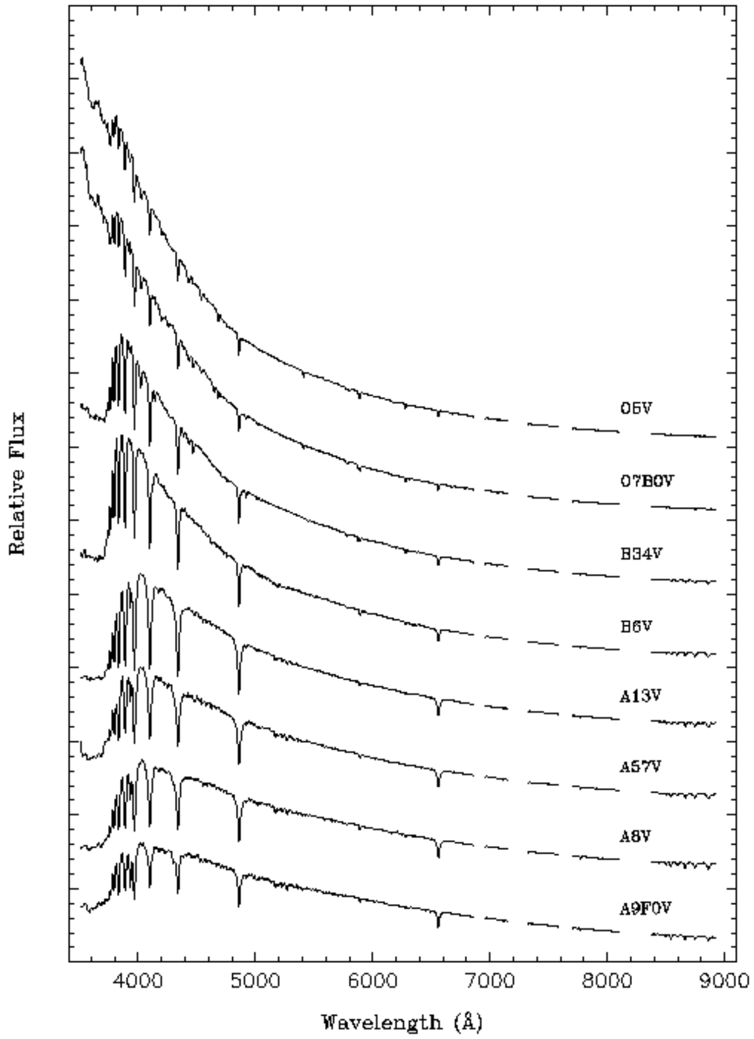
T < 5,000K: all Hydrogen is in ground (n=1) state ➔ no lines

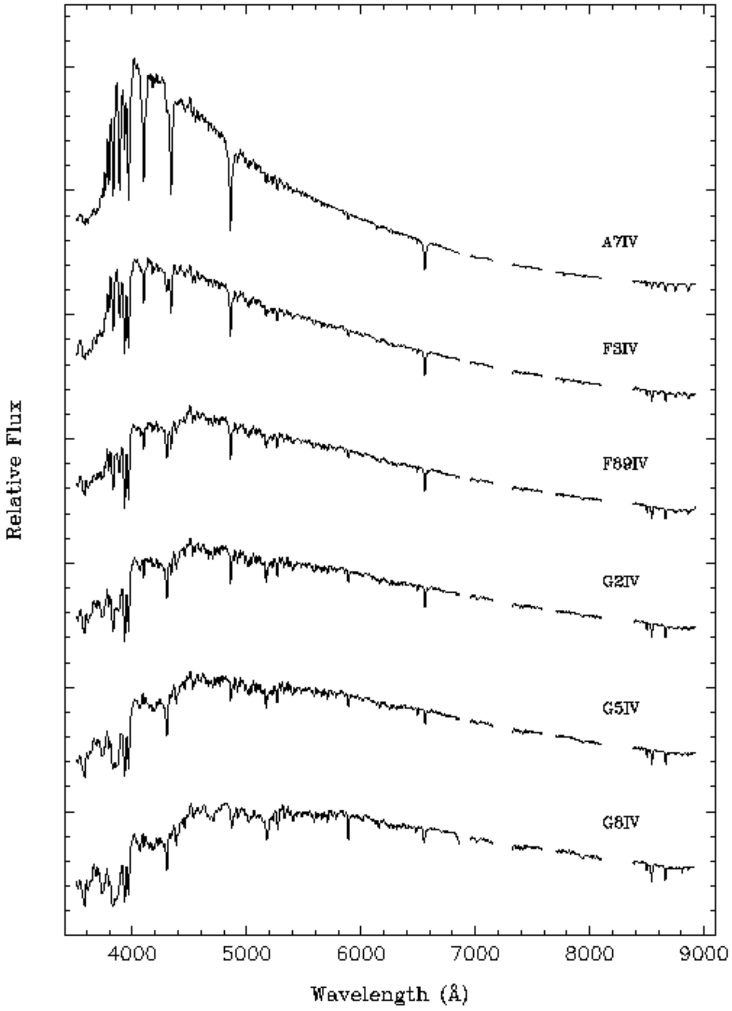
T > 20,000K: all Hydrogen is ionized ➔ no lines

T ~ 10,000K: some Hydrogen is in n=2 state ➔ strong lines

- Spectral lines are observational indicators of T_e

Stellar spectra





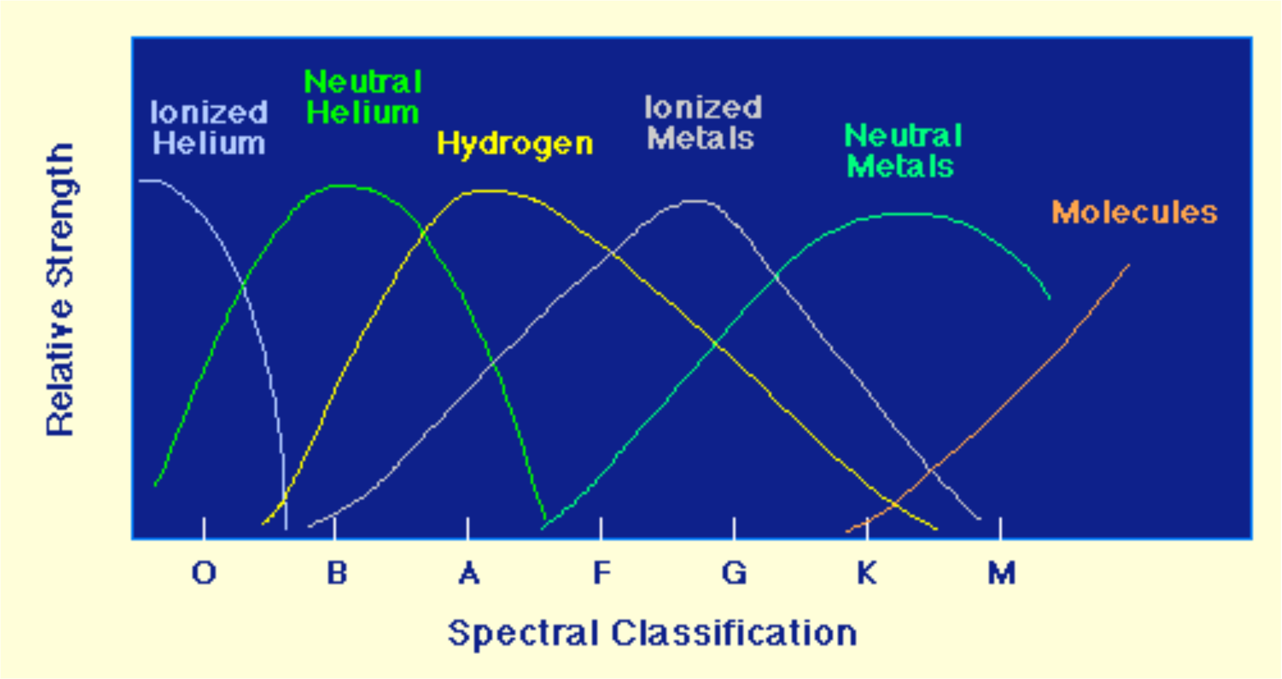
Spectral lines

Lines depend on temperature in stellar atmosphere
+
ionization potentials for relevant species

e.g.,	H	HeI	HeII	CaI	CaII	FeI
	13.6eV	24.6eV	54.5eV	6.1eV	11.9eV	7.9eV

Ionization occurs when kT ~ ionization potential/10

Spectral classification

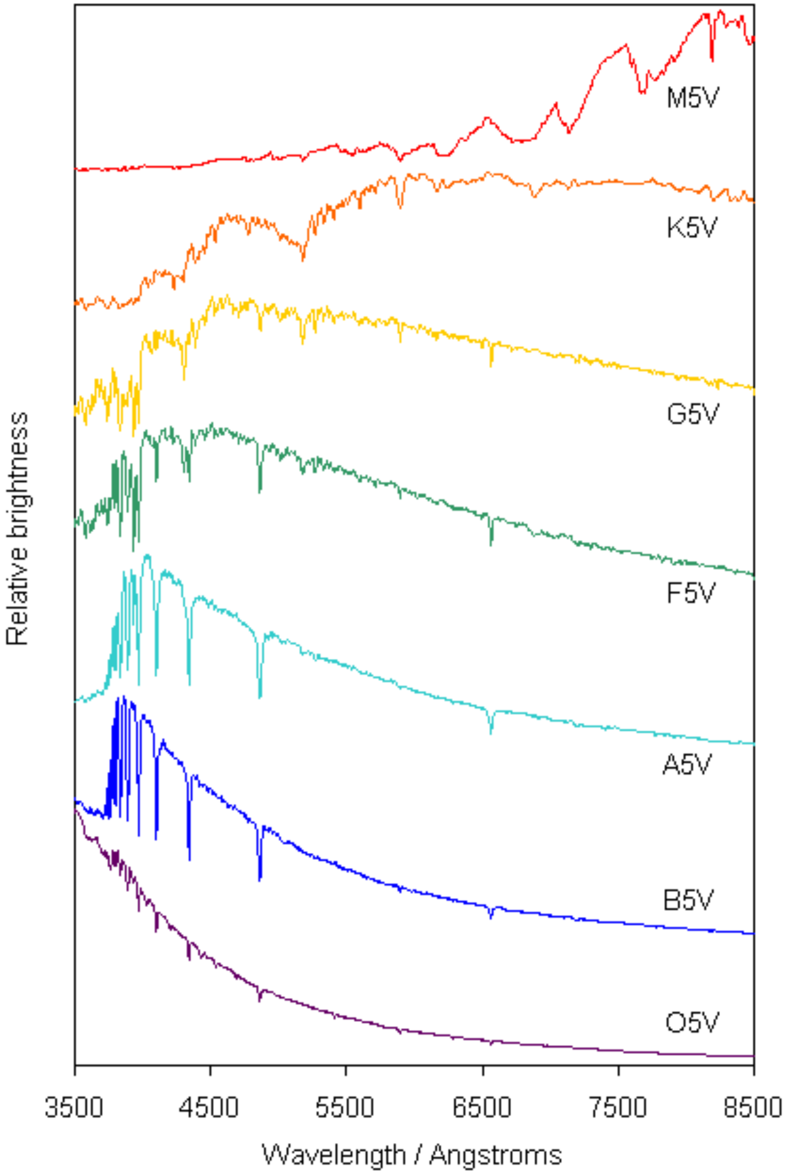


T_e: 40K 20K 10K 6.7K 5.5K 4.5K 3.5K

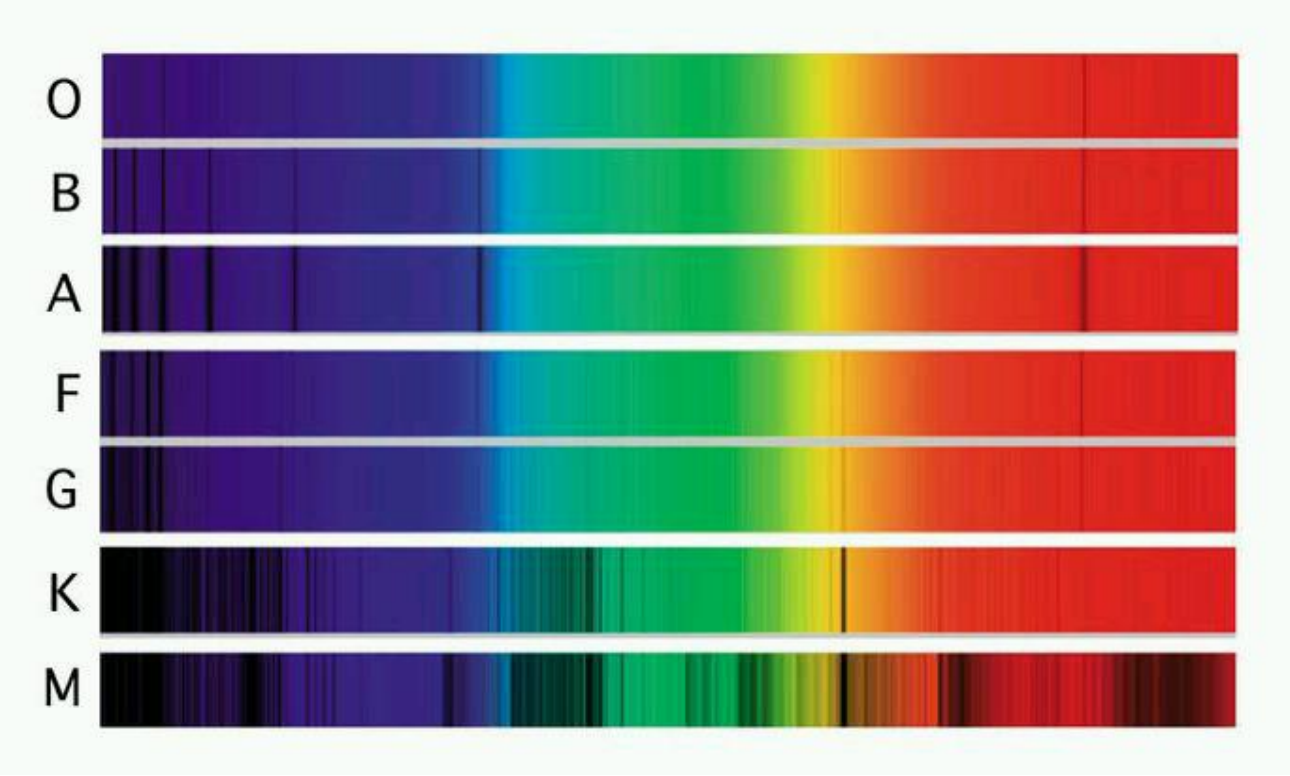
early type

late type

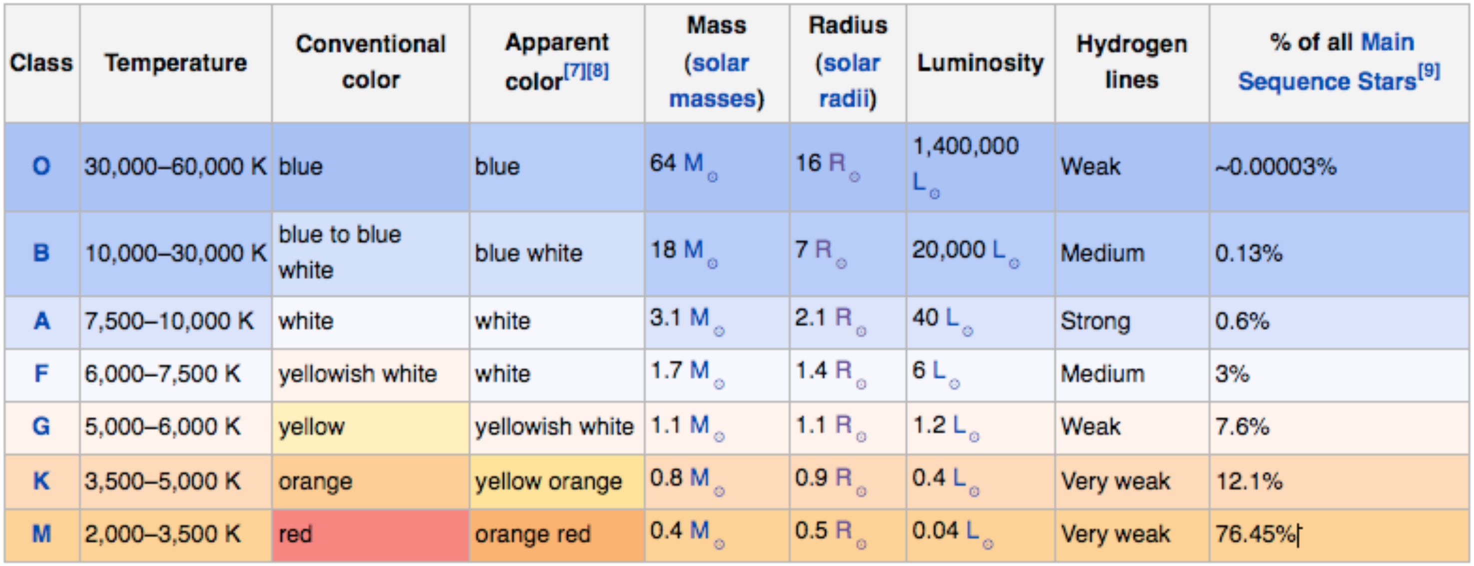
Spectral classification



Spectral classification



Spectral classification



Class	Temperature	Conventional color	Apparent color[7][8]	Mass (solar masses)	Radius (solar radii)	Luminosity	Hydrogen lines	% of all Main Sequence Stars[9]
O	30,000–60,000 K	blue	blue	64 $M_\odot$	16 $R_\odot$	1,400,000 $L_\odot$	Weak	~0.00003%
B	10,000–30,000 K	blue to blue white	blue white	18 $M_\odot$	7 $R_\odot$	20,000 $L_\odot$	Medium	0.13%
A	7,500–10,000 K	white	white	3.1 $M_\odot$	2.1 $R_\odot$	40 $L_\odot$	Strong	0.6%
F	6,000–7,500 K	yellowish white	white	1.7 $M_\odot$	1.4 $R_\odot$	6 $L_\odot$	Medium	3%
G	5,000–6,000 K	yellow	yellowish white	1.1 $M_\odot$	1.1 $R_\odot$	1.2 $L_\odot$	Weak	7.6%
K	3,500–5,000 K	orange	yellow orange	0.8 $M_\odot$	0.9 $R_\odot$	0.4 $L_\odot$	Very weak	12.1%
M	2,000–3,500 K	red	orange red	0.4 $M_\odot$	0.5 $R_\odot$	0.04 $L_\odot$	Very weak	76.45%

Oh Be A Fine Girl/Guy Kiss Me

Omnivorous Butchers Always Find Good Kangaroo Meat

Only Bored Astronomers Find Gratification Knowing Mnemonics

Luminosity class

Stars of same type have different line widths

Same T, different R → different surface gravity g →
→ different surface pressure P

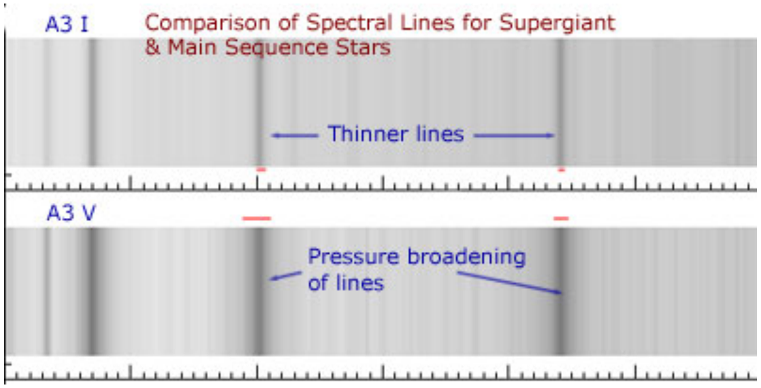
Pressure broadening: orbitals of atoms are perturbed due to collisions → broadening of spectral lines.

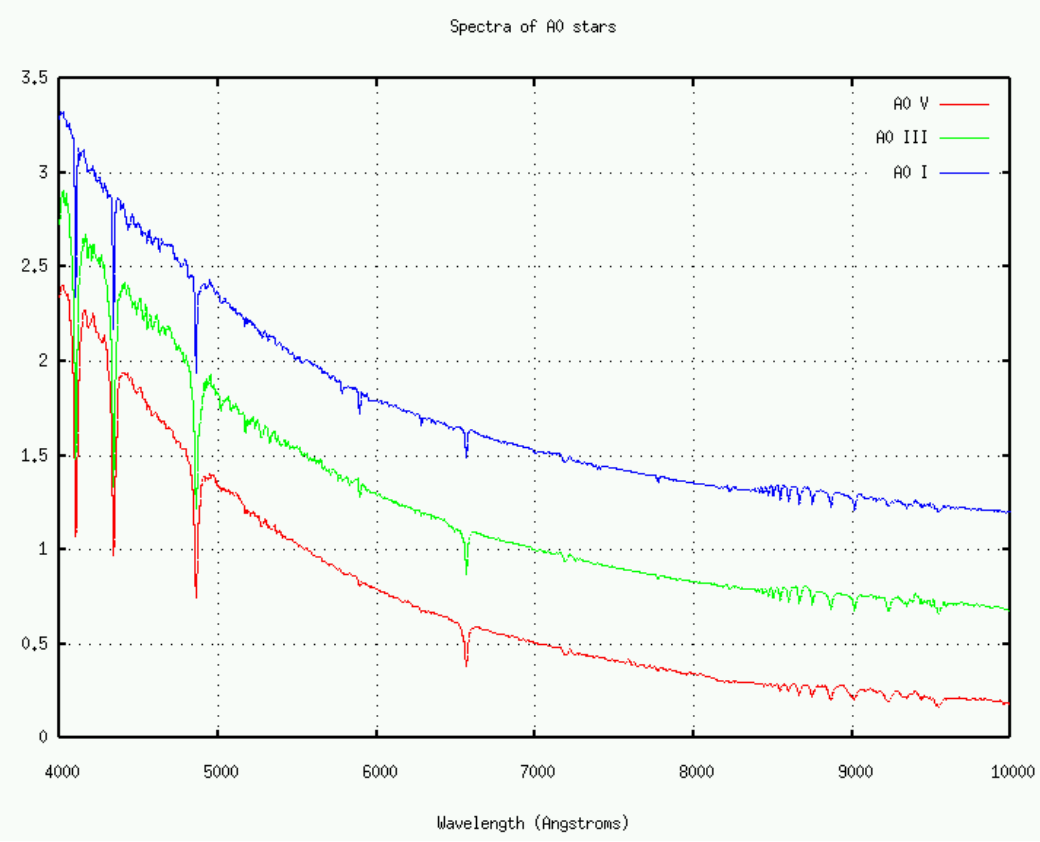
Since $L = 4\pi R^2 \sigma_T T^4$, changes in R at fixed T are changes in L

Spectral line widths → luminosity classification

I_a	I_b	II	III	IV	V	VI	VII
Supergiants		Luminous giants	Giants	Subgiants	Dwarfs	Subdwarfs	White dwarfs

Luminosity class





Special stars

C: carbon stars - same T_{eff} as K, M stars, but higher abundance of C than O → all O goes to form CO. Remaining C forms C2, CN.

S: same T_{eff} as K, M stars, but have extra heavy elements

W: Wolf-Rayet - He in atmosphere instead of H, strong winds

L: cooler than M stars. Some do not have fusion.

T: cool brown dwarfs (700-1,000K). Methane lines are prominent.

The Sun is a G2V star

The first Hertzsprung-Russell (H-R) diagram

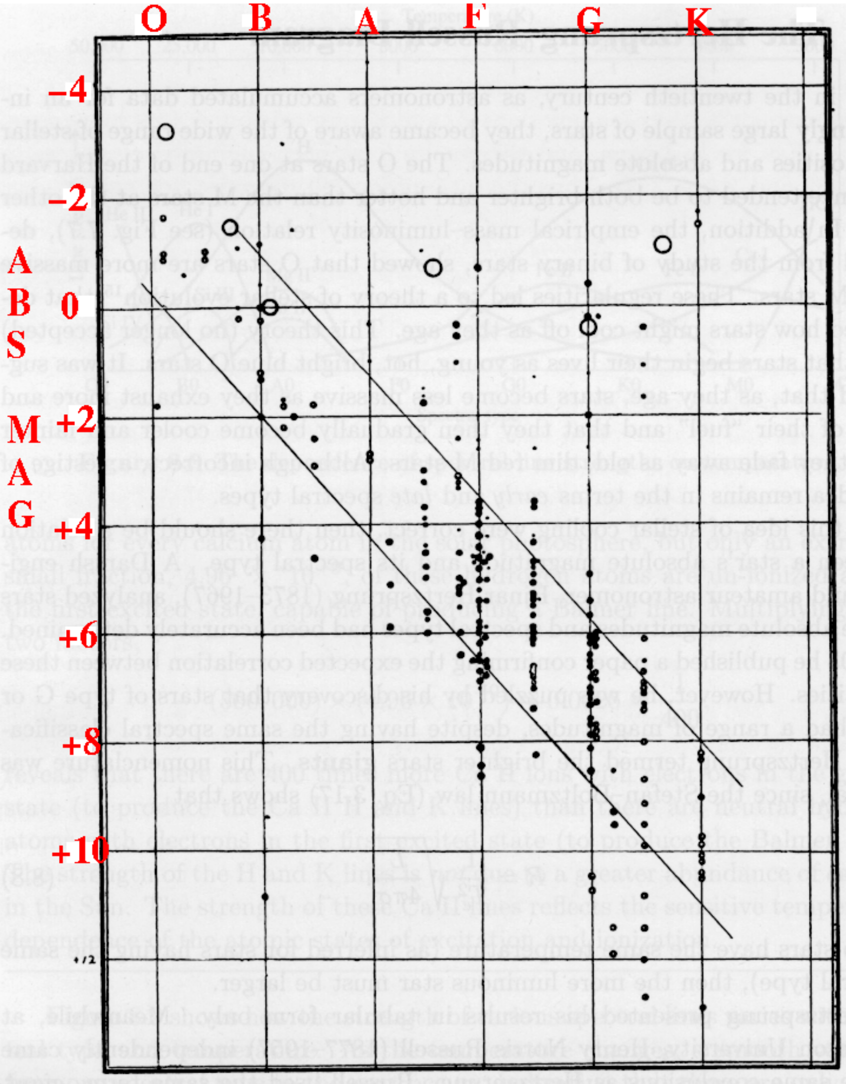
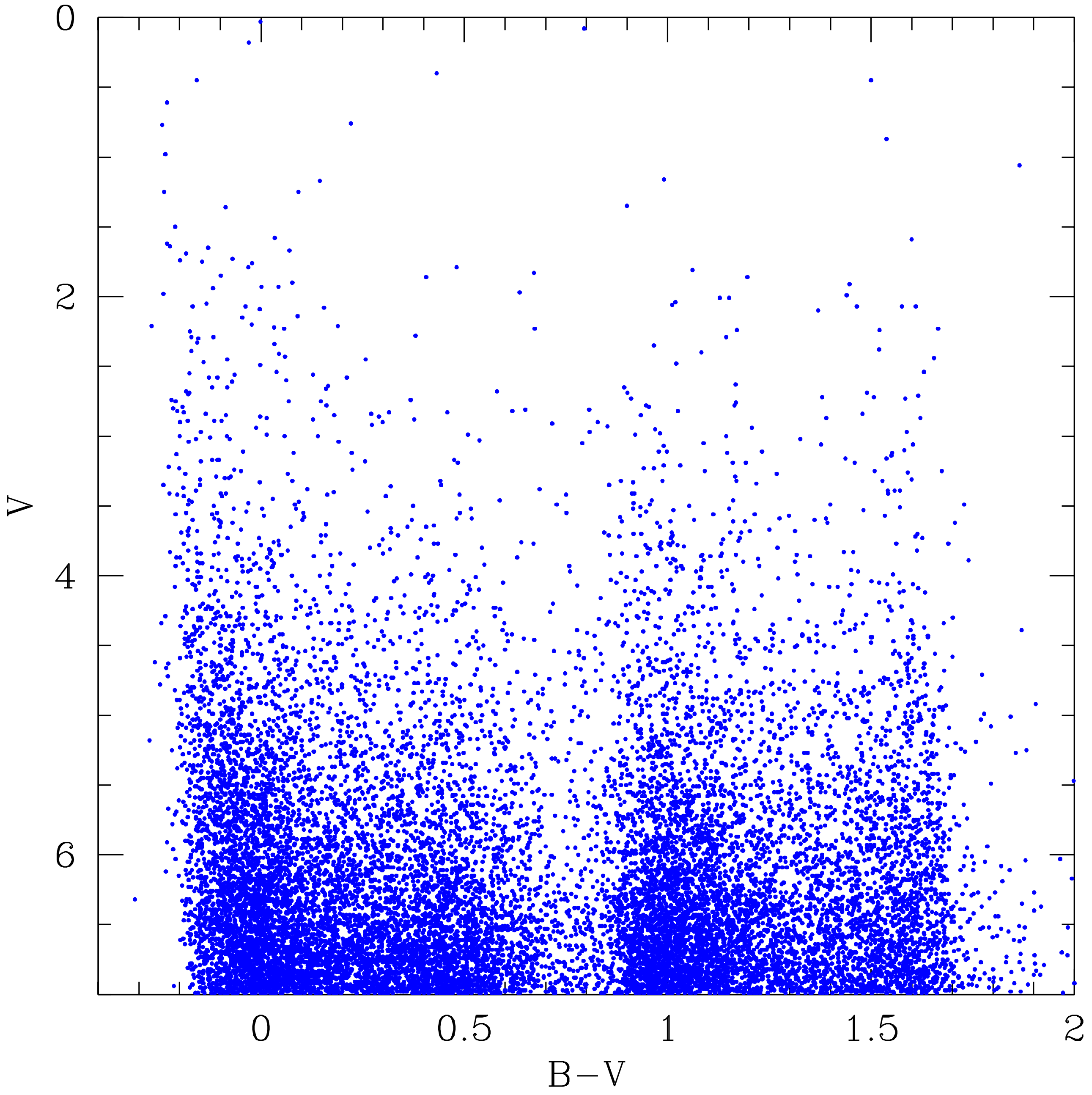
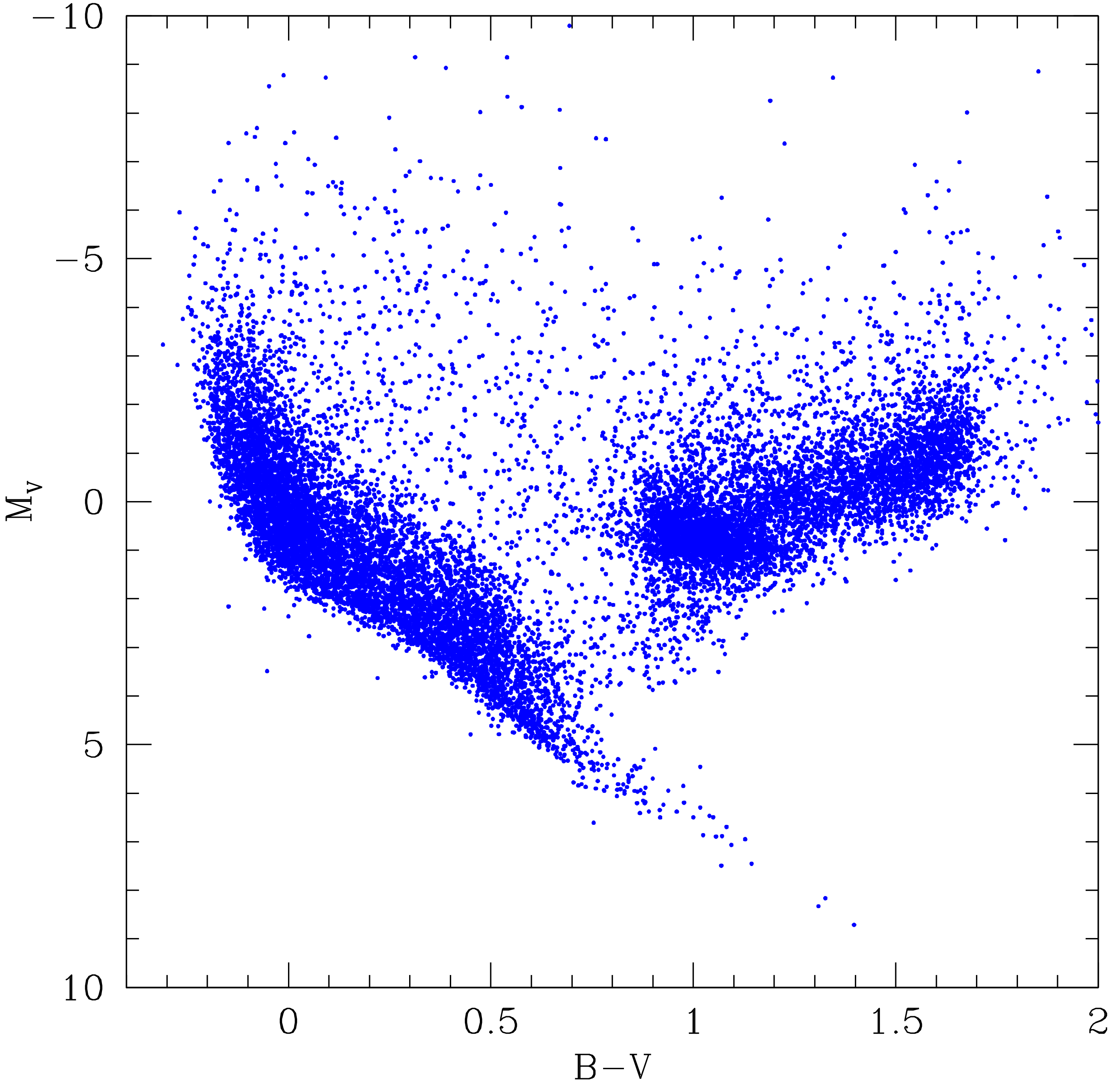


Figure 8.10 Henry Norris Russell's first diagram, with spectral types listed along the top and absolute magnitudes on the left-hand side. (Figure from Russell, *Nature*, *93*, 252, 1914.)

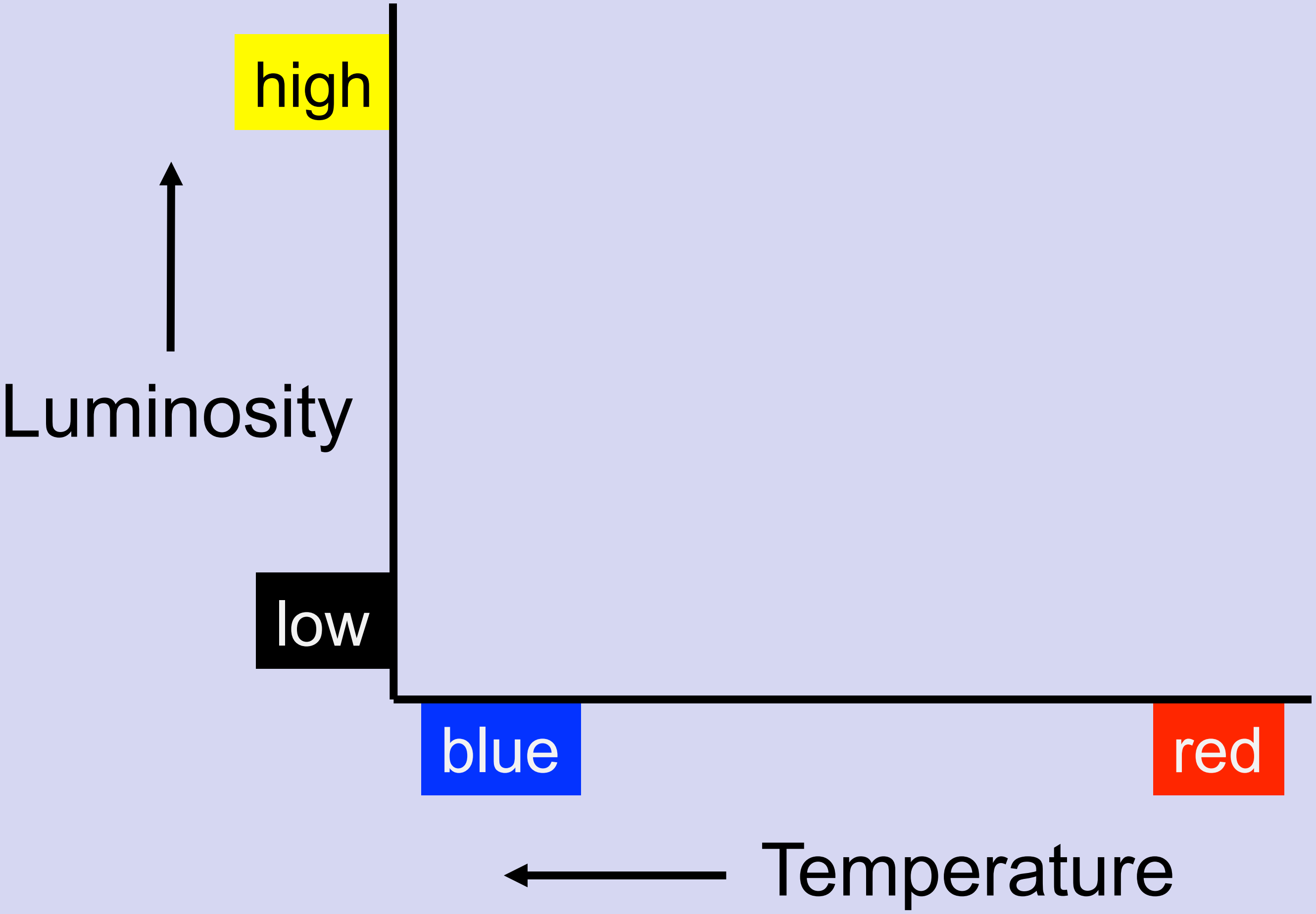
Hipparcos Color-Magnitude Diagram



Hipparcos Color-Magnitude Diagram



Plot luminosity vs. temperature



high

Luminosity

low

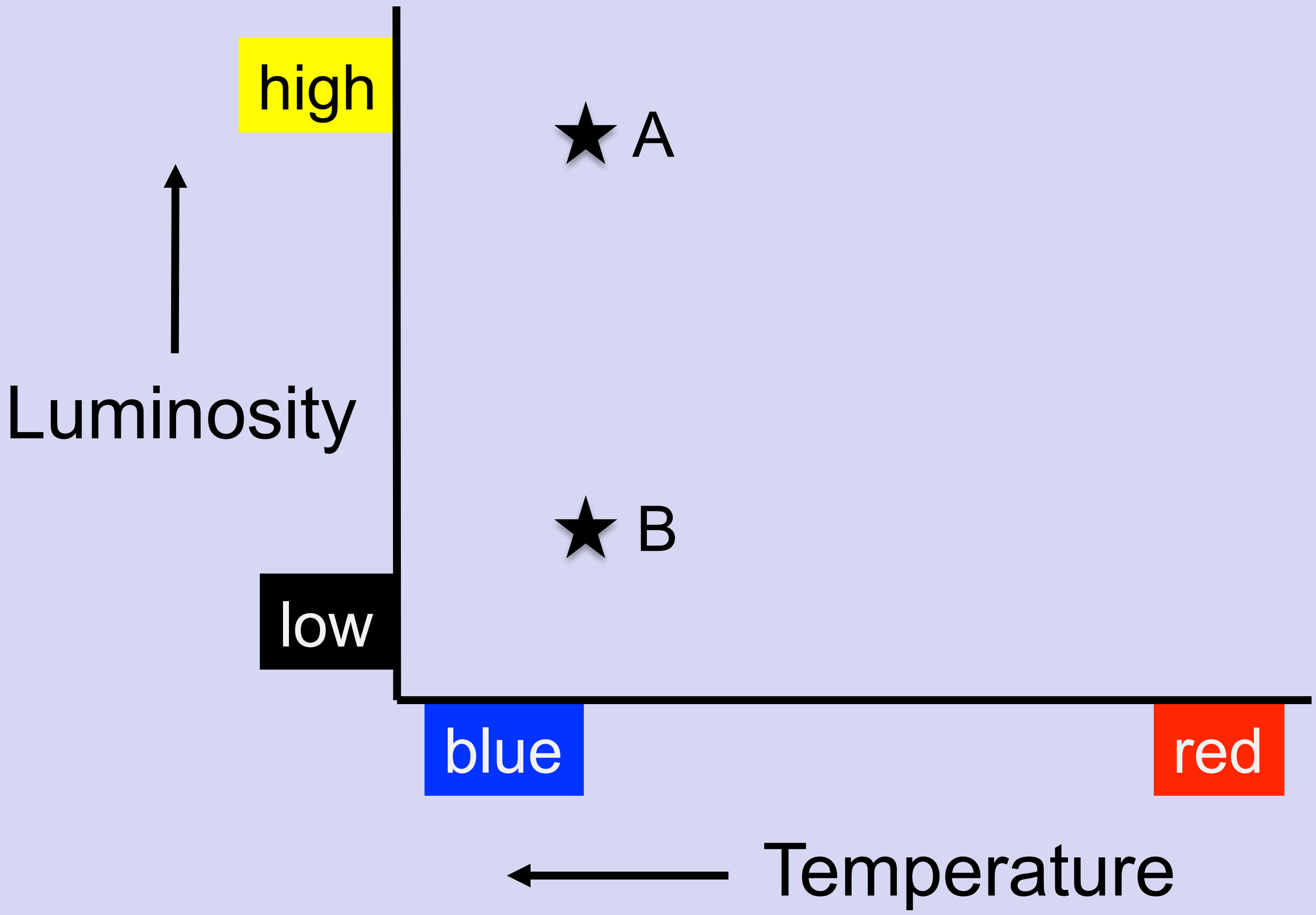
blue

red

Temperature

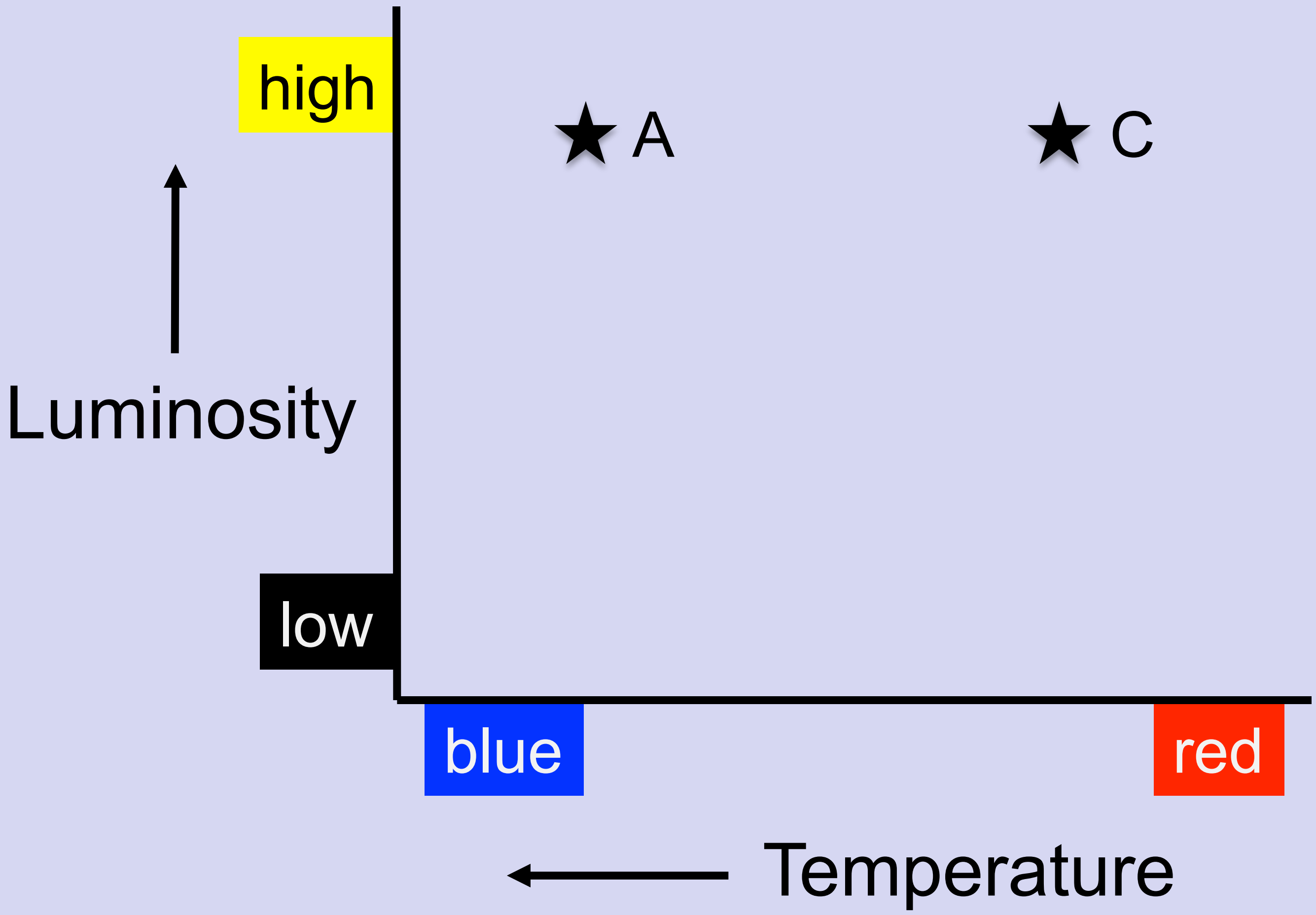
Understanding the H-R diagram

$$L = 4\pi R^2 \sigma T^4$$



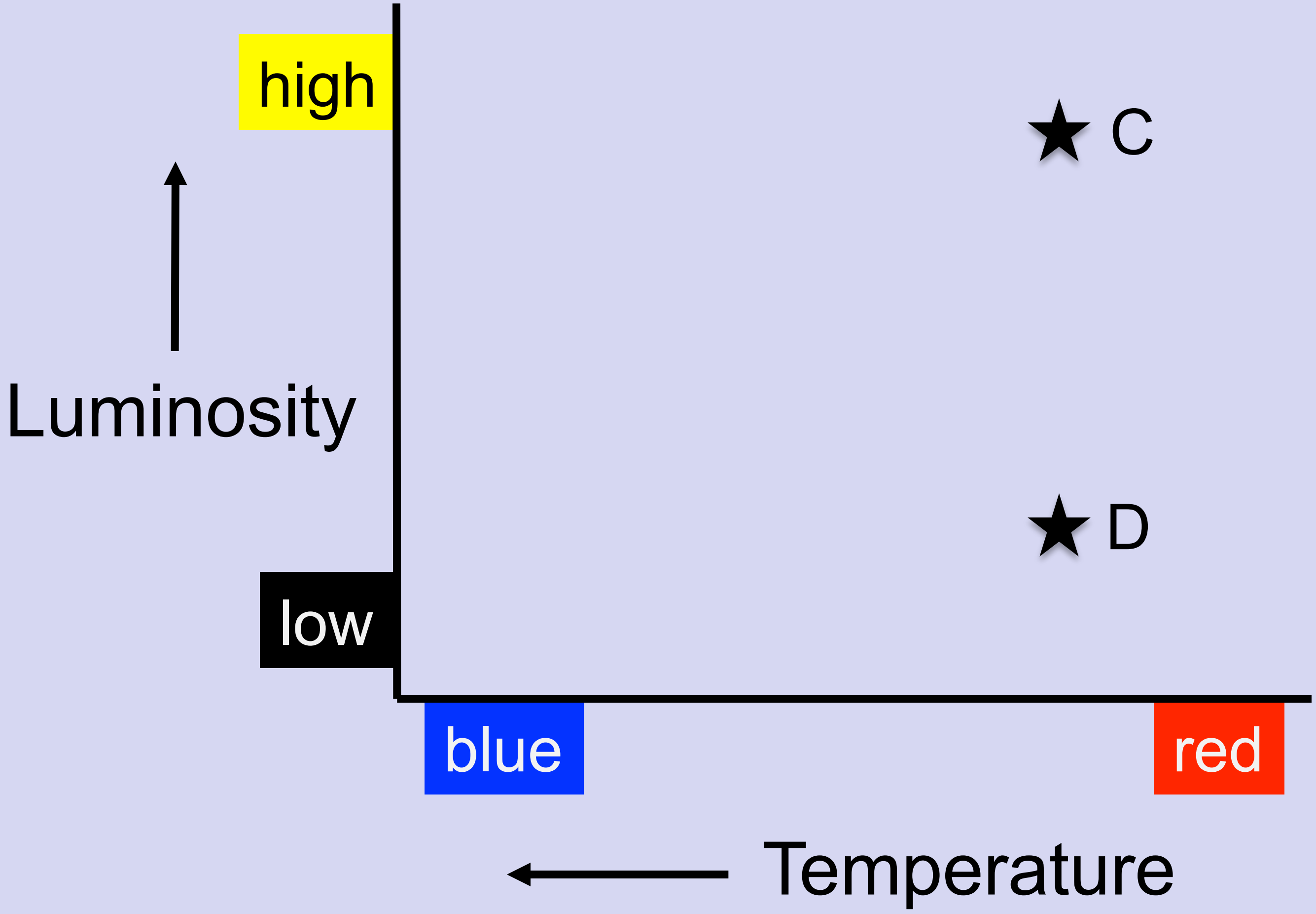
Understanding the H-R diagram

$$L = 4\pi R^2 \sigma T^4$$



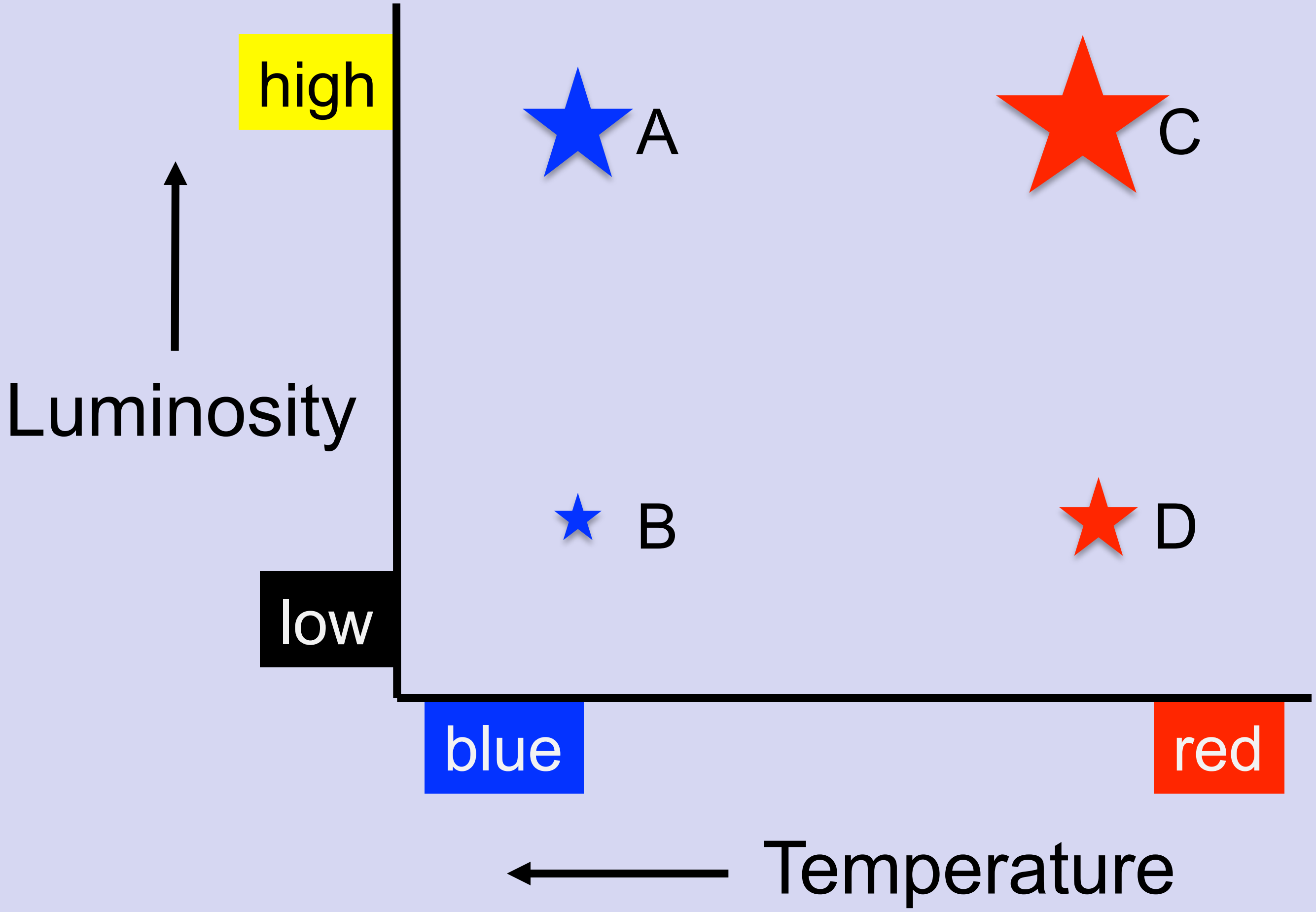
Understanding the H-R diagram

$$L = 4\pi R^2 \sigma T^4$$



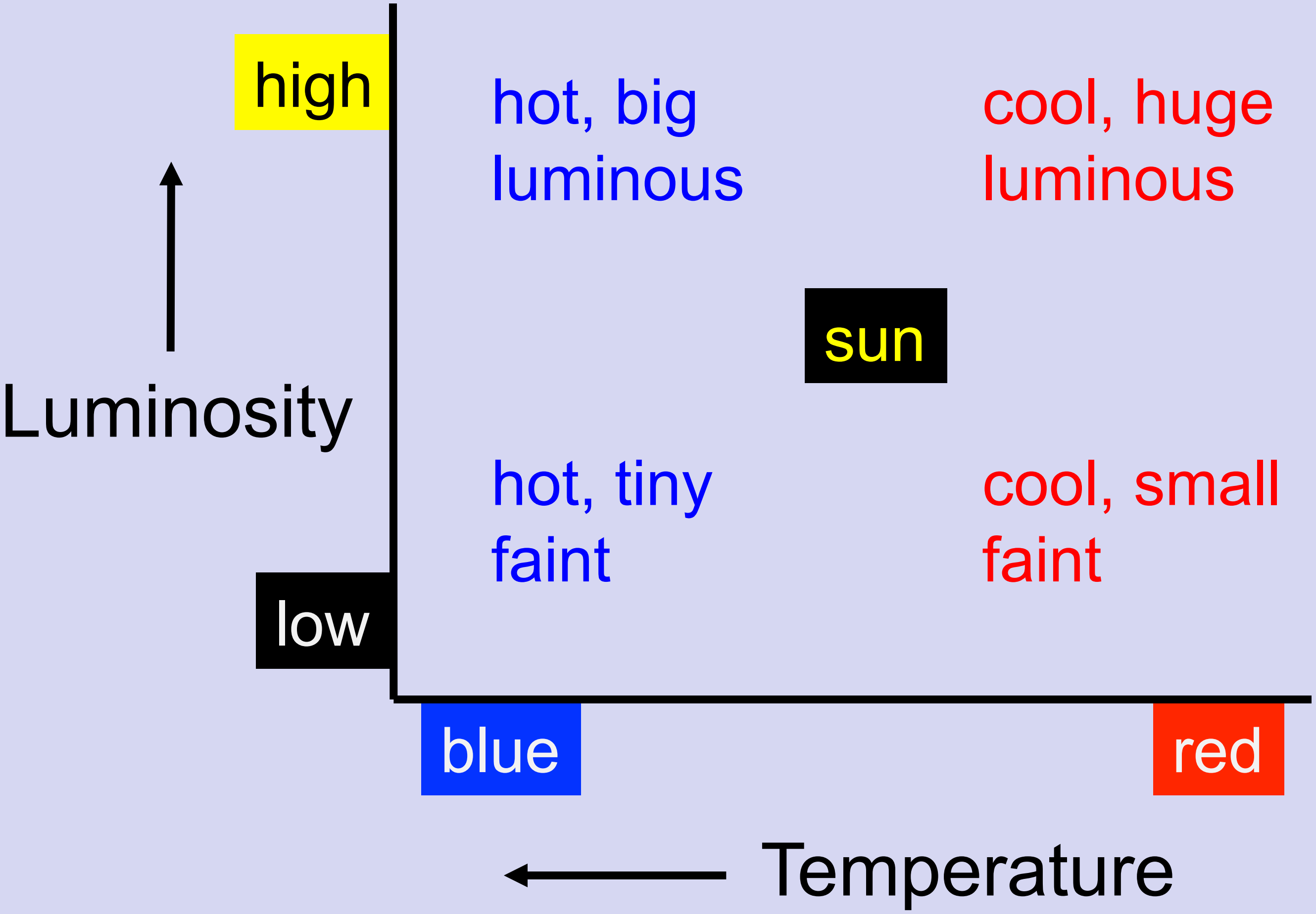
Understanding the H-R diagram

$$L = 4\pi R^2 \sigma T^4$$

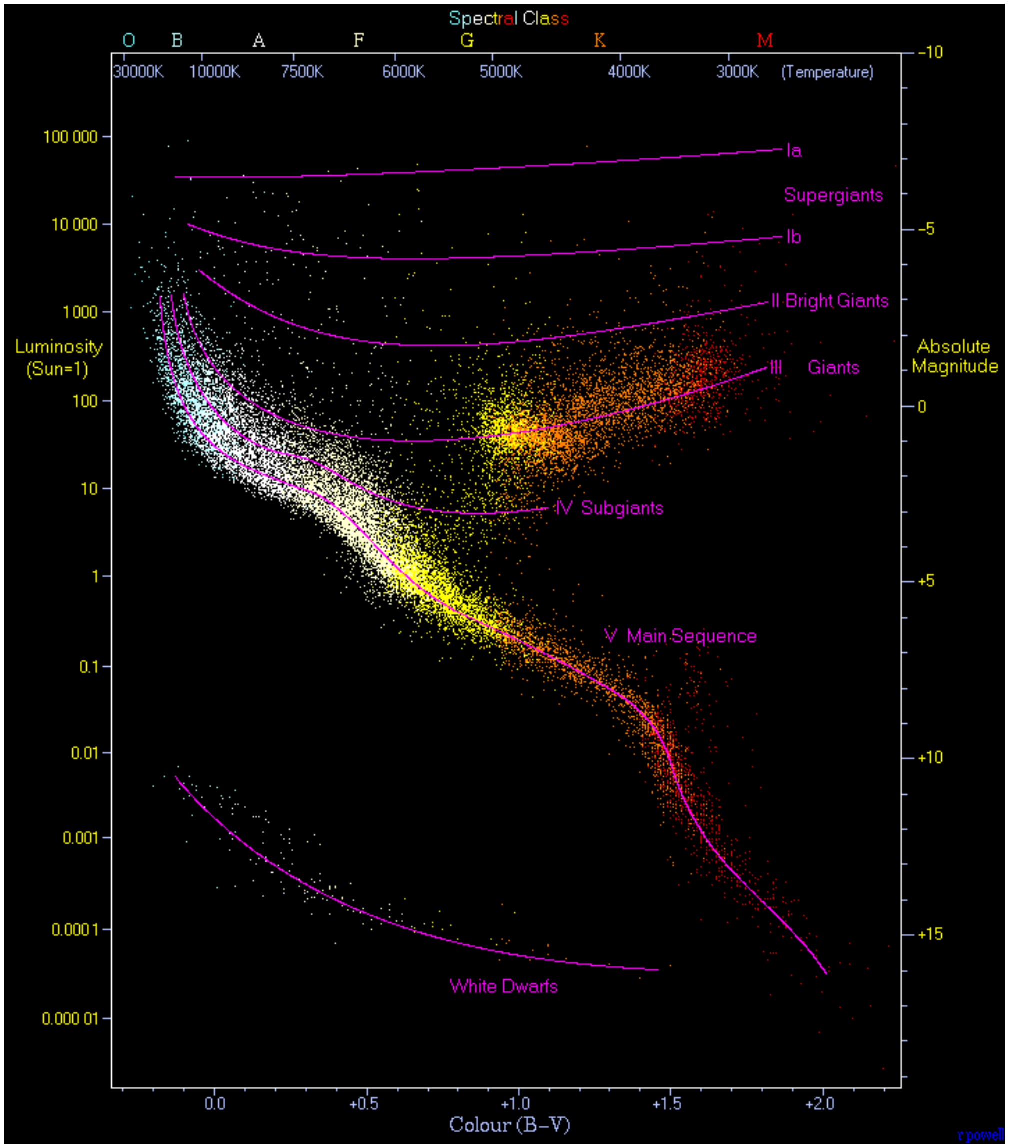


Understanding the H-R diagram

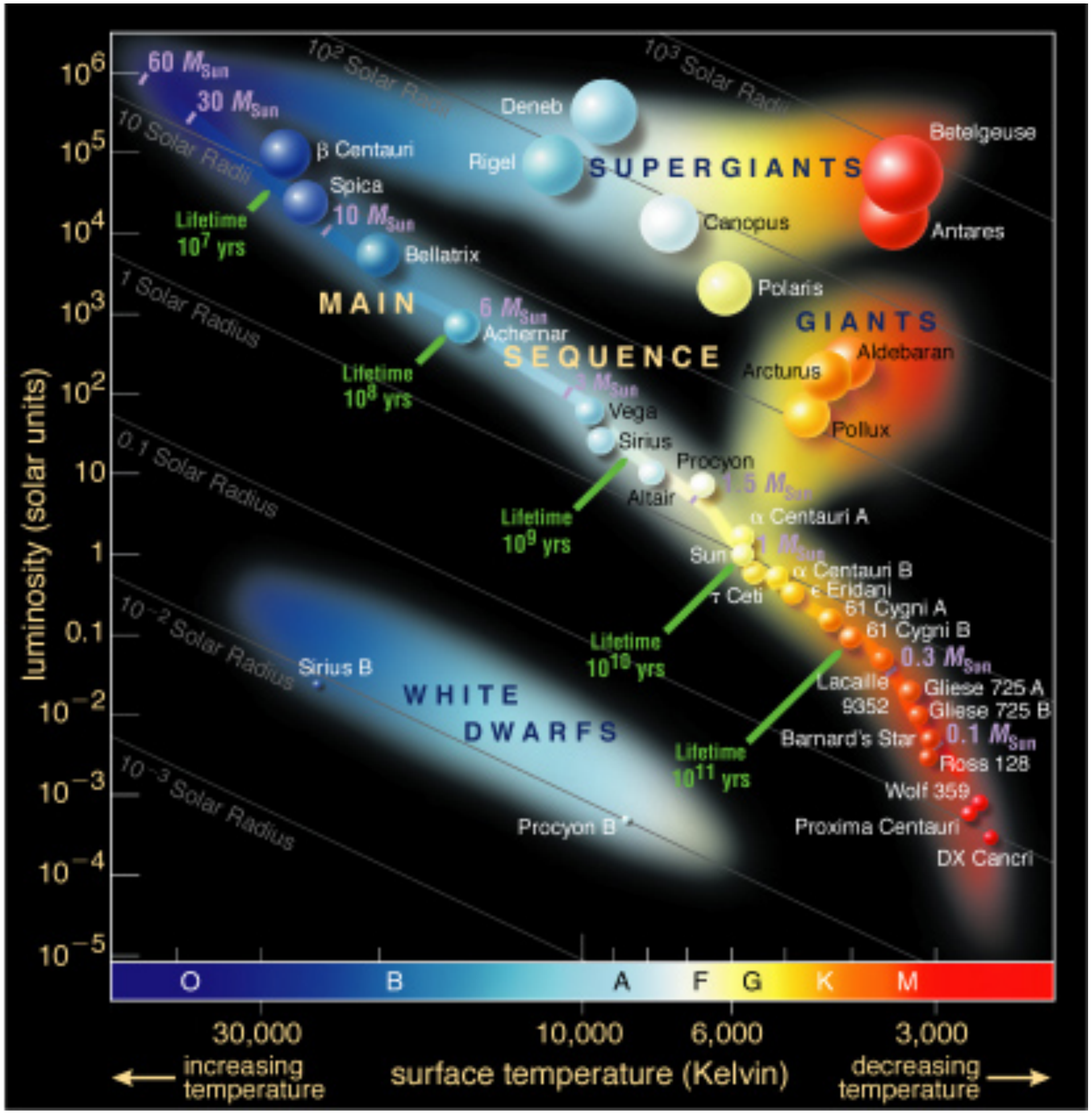
$$L = 4\pi R^2 \sigma T^4$$



Hipparcos H-R diagram



Theoretical H-R diagram



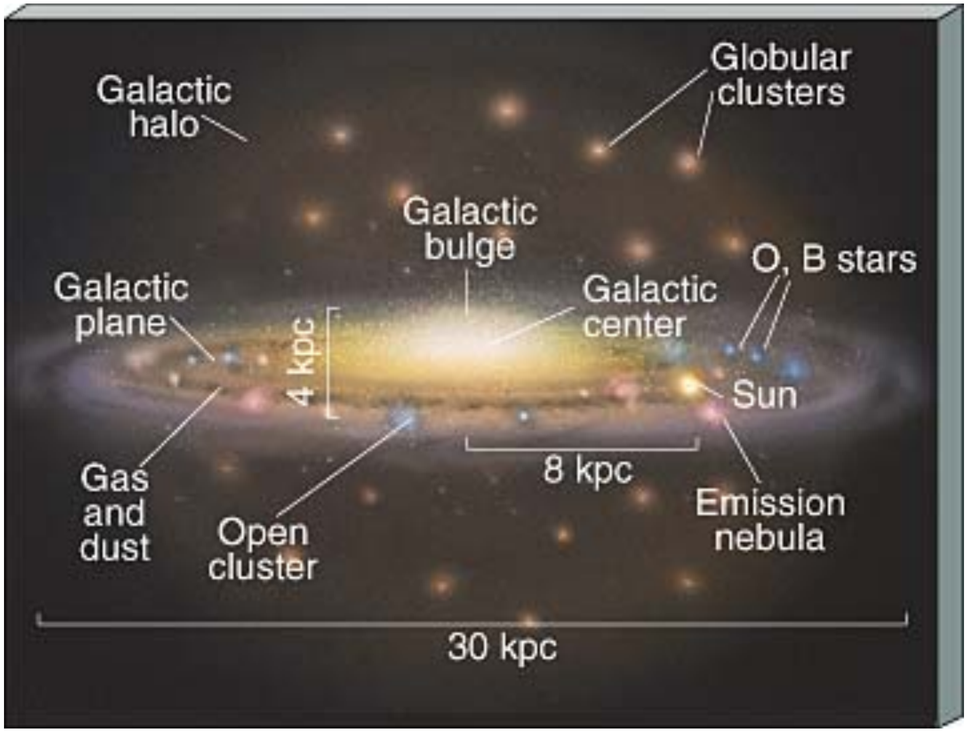
Chemical composition

Primordial (Big Bang) nucleosynthesis: protons fuse to form He and heavier elements 3 minutes after the Big Bang. Ends 20 minutes later.
Alpher, Bethe & Gammow, Physical Review L, 1948

75% H 25% He 0.01% D

Subsequent fusion inside massive stars and enrichment of the inter-stellar medium via supernovae, leads to future generations of stars with more heavy elements.

Stellar populations in the Milky Way



Stellar populations in the Milky Way

	Pop I	Pop II	Pop III
Spatial Distribution	Disk, |z|<200pc	Halo/spheroid	Have not been found
Kinematics (coherent)	Disk rotation (220 km/s)	No rotation	
Kinematics (dispersion)	~30 km/s	large	
Metallicity	Z~0.02	Z<0.01	Z~0
Age	Young	Old	Primordial

Determining stellar properties

Measurements:

- position on sky
- flux in different bands
- spectrum
- time dependence of above

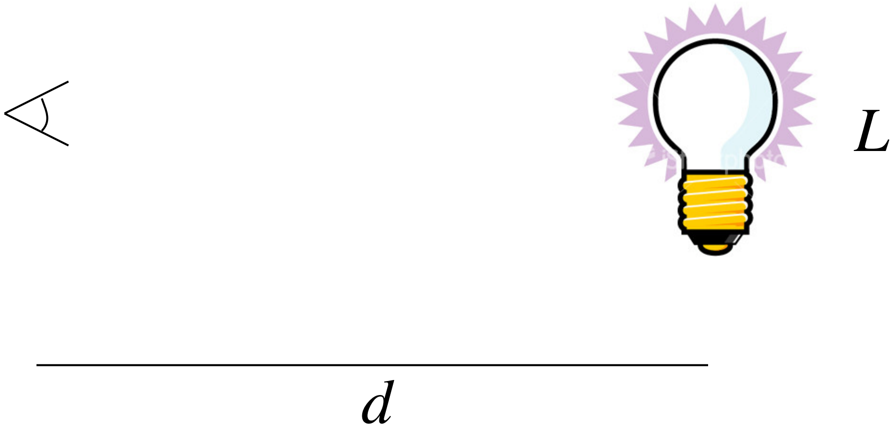
Physical parameters:

• distance	d
• luminosity	L
• temperature	T_e
• radius	R
• mass	M
• age	t
• chemical composition	X_i

Determining distance

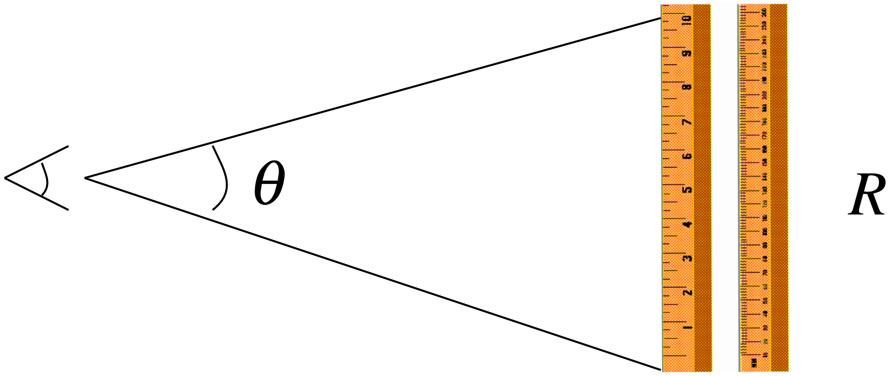
Standard candle

$$d = \left(\frac{L}{4\pi f} \right)^{\frac{1}{2}}$$



Standard ruler

$$d = \frac{R}{\theta}$$

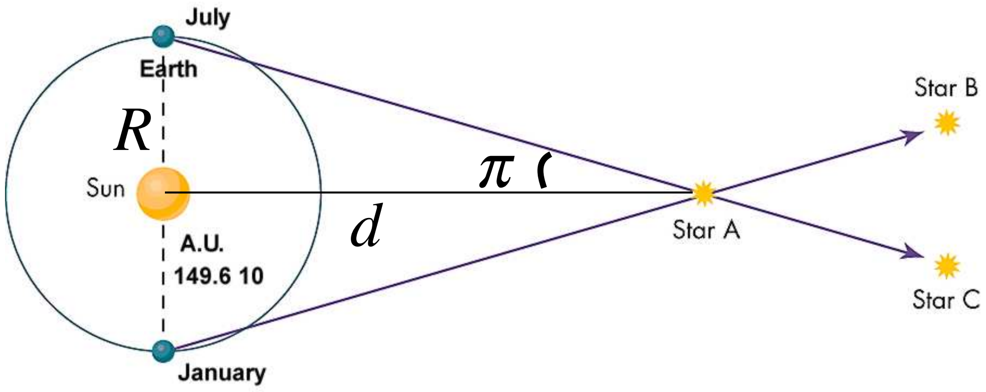


Determining distance: Parallax

RULER

$$\tan\pi = \frac{R}{d} \approx \pi$$

$$R = 1AU = 1.5\times 10^{13}\,cm$$

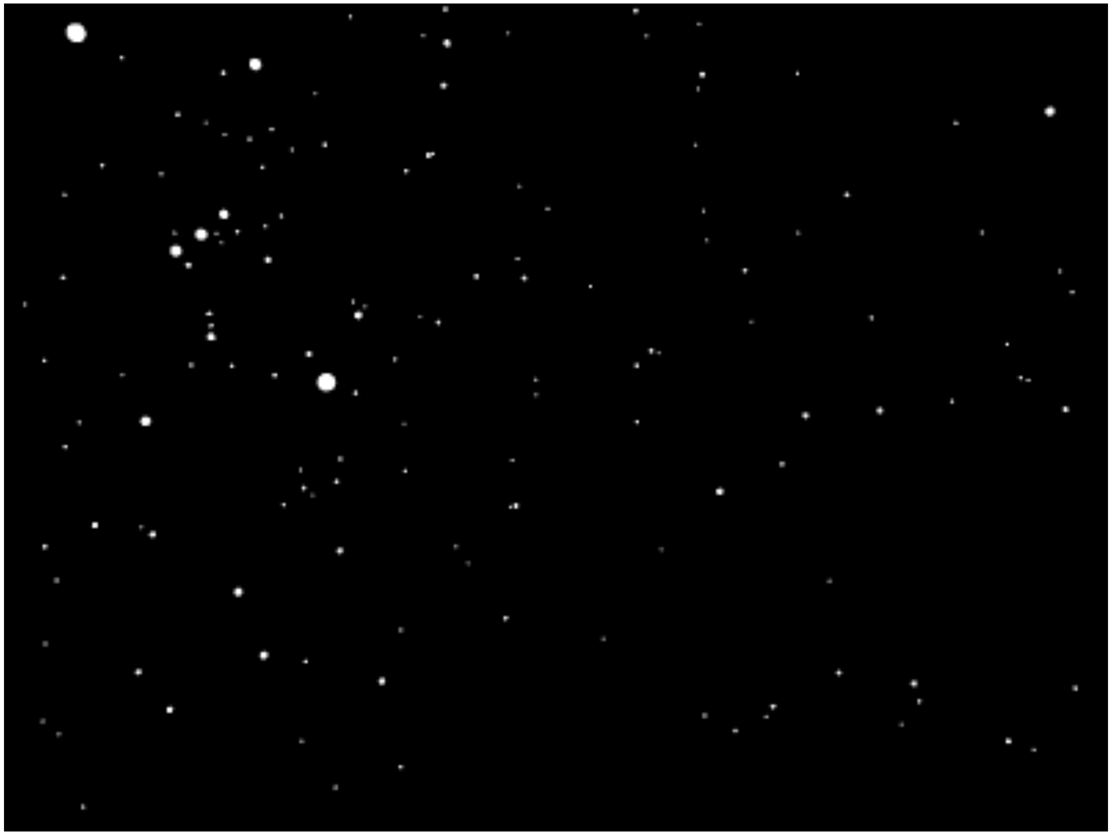


Define new distance unit: parsec (**par**allax-**sec**ond)

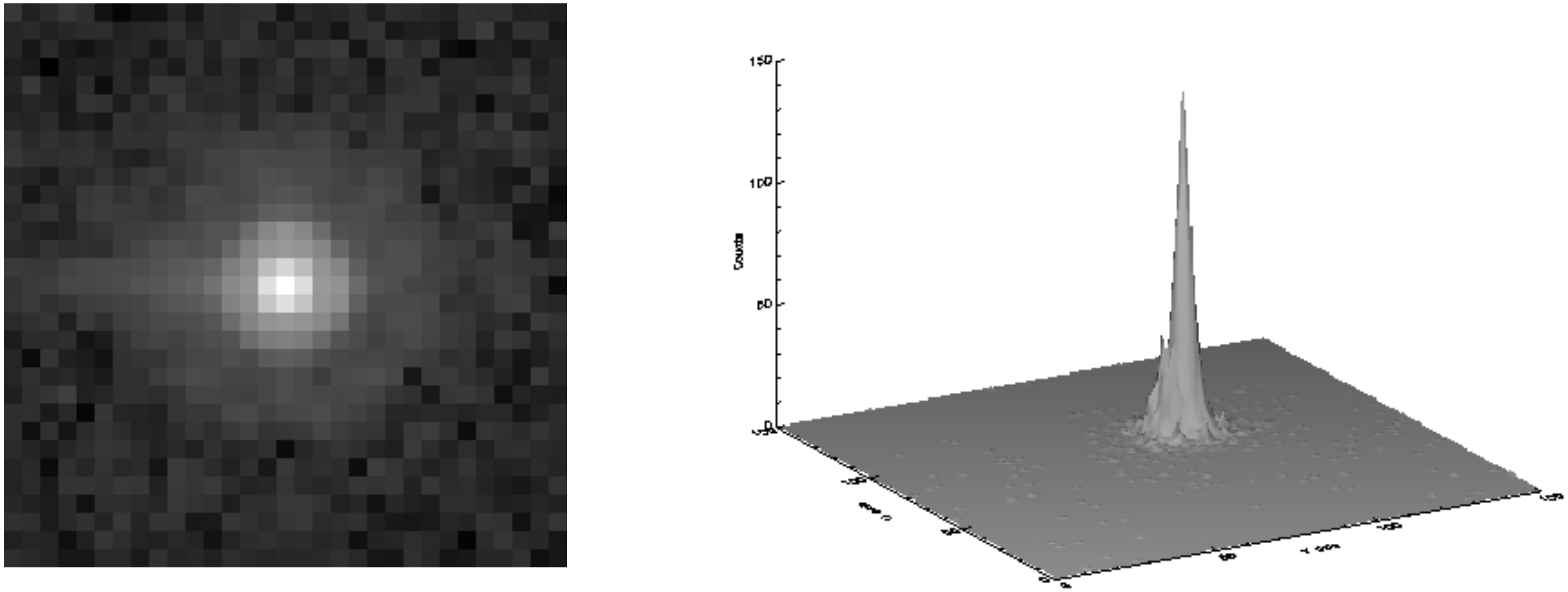
$$1\,pc = \frac{1AU}{\tan(1'')} = 206{,}265\,AU = 3.26\,ly$$

$$\boxed{\left(\frac{d}{1\,pc}\right) = \frac{1}{\pi''}}$$

Determining distance: Parallax



Determining distance: Parallax



Point spread function (PSF)

Determining distance: Parallax

Need high angular precision to probe far away stars.

$$\frac{\sigma_d}{d} = \frac{\sigma_\pi}{\pi} = d\sigma_\pi \rightarrow d = \left(\frac{\sigma_d}{d}\right)\frac{1}{\sigma_\pi}$$

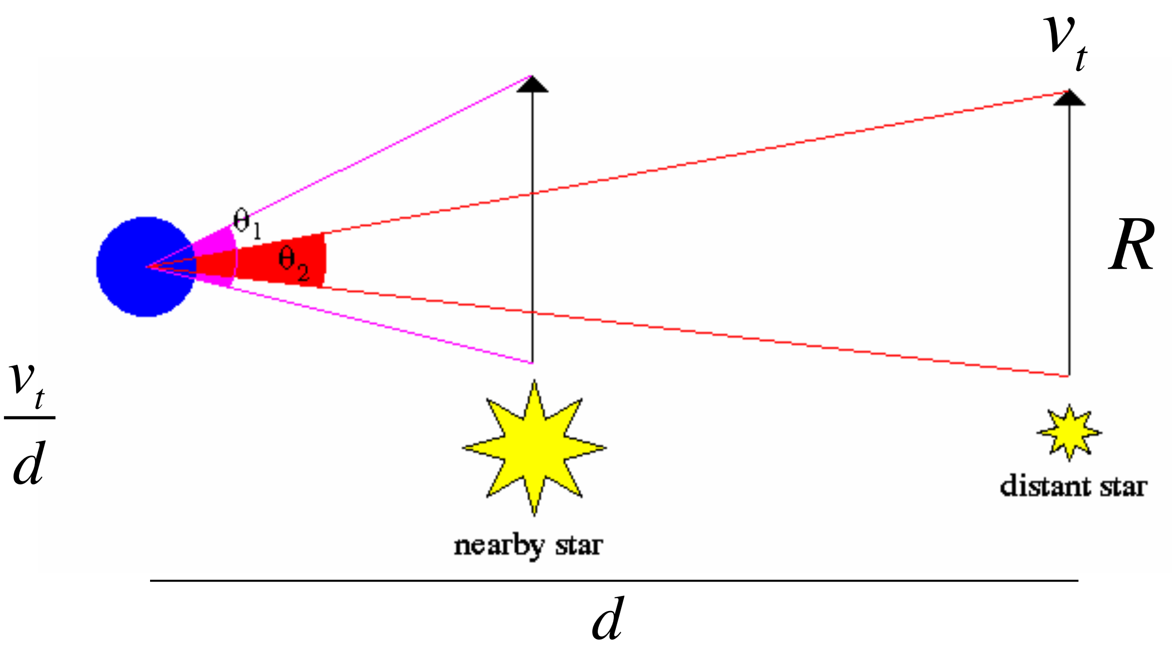
e.g., to get 10% distance errors

$$d_{\max} = \frac{0.1}{\sigma_\pi}$$

Mission	Dates	σ_π	$d_{\max}$
Earth telescope		$\sim 0.1\ as$	$1\ pc$
Hipparcos	1989-1993	$\sim 1\ mas$	$100\ pc$
Gaia	2013-2018	$\sim 20\ \mu as$	$5\ kpc$
SIM	cancelled	$\sim 4\ \mu as$	$25\ kpc$

Determining distance: moving cluster method

Proper motion

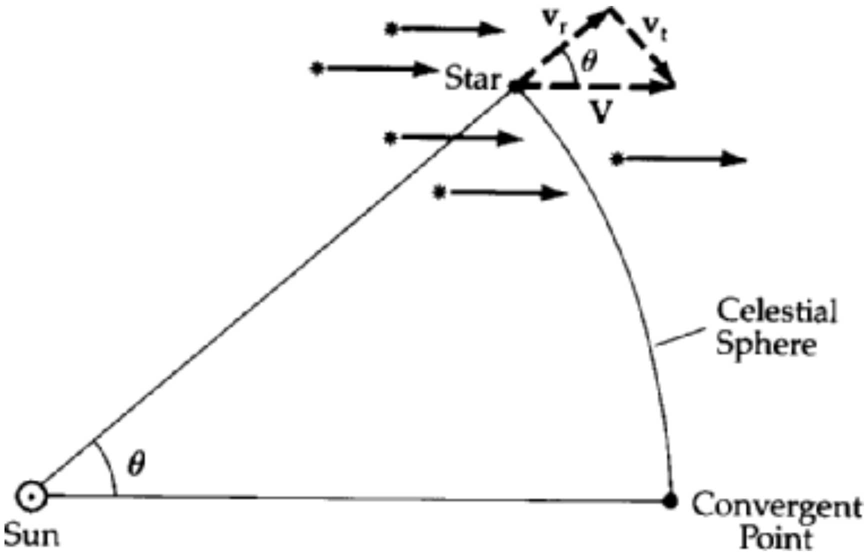


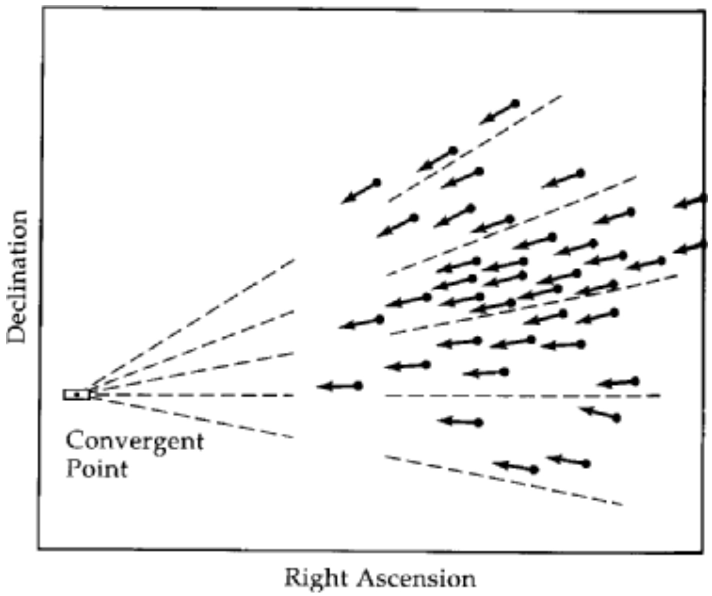
$$\mu = \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{R}{d}\right) = \frac{v_t}{d}$$

$$\boxed{\left(\frac{d}{1pc}\right) = \frac{\left(v_t/1\mathrm{kms}^{-1}\right)}{4.74\left(\mu/1''\mathrm{yr}^{-1}\right)}}$$

Determining distance: moving cluster method

RULER





1. Measure proper motions of stars in a cluster
2. Obtain convergent point
3. Measure angle between cluster and convergent point θ
4. Measure radial velocity of cluster v_r
5. Compute tangential velocity v_t
6. Use proper motion of cluster to get distance d

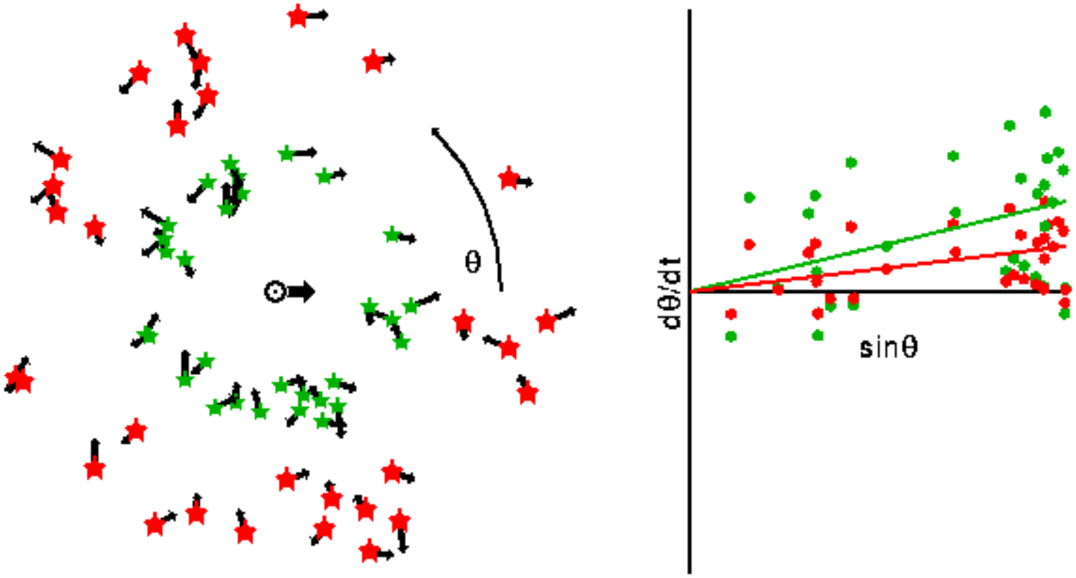
$$d = \frac{v_r \tan\theta}{4.74\mu}$$

Determining distance: secular parallax

- Parallax method is limited by 2AU baseline of earth’s orbit
- Sun moves ~4AU/yr toward Vega relative to local rotation of Galactic disk
- Over a few years, this can build up to a large baseline
- Unfortunately, other stars are not at rest, rather have unknown motions
- However, if we average over many stars, their mean motion should be zero (relative to local rotation)
- Can therefore get the mean distance to a set of stars

Determining distance: secular parallax

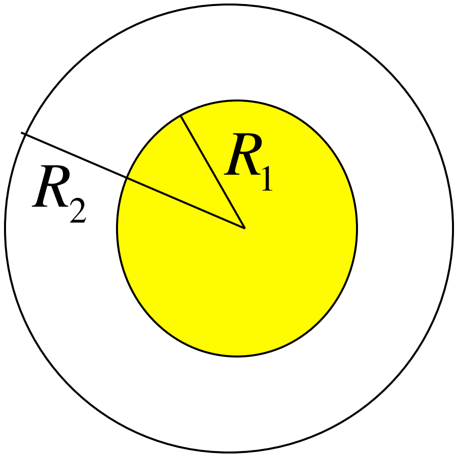
RULER



Determining distance: Baade-Wesselink (moving stellar atmosphere)

CANDLE

For pulsating stars, SN, novae, measure flux and effective temperature at two epochs, as well as the radial velocity and time between the epochs.



$$f_1, f_2, T_1, T_2, v_r(t), \Delta t$$

$$\frac{f_1}{f_2} = \frac{L_1}{L_2} = \left(\frac{R_1}{R_2}\right)^2 \left(\frac{T_1}{T_2}\right)^4 \rightarrow \frac{R_1}{R_2} = \left(\frac{f_1}{f_2}\right)^{\frac{1}{2}} \left(\frac{T_2}{T_1}\right)^2$$

$$R_2 - R_1 = \overline{v}_r \times \Delta t$$

Solve for R_1 and R_2. R and T $\longrightarrow$ L $\longrightarrow$ d

Determining distance: spectroscopic parallax

CANDLE

- Stellar spectra alone can give us the luminosity class + spectral class ➔ position on HR diagram

- This gives the absolute magnitude, which gives the distance modulus.

- Basically, compare a star's spectrum to an identical spectrum of another star with known luminosity.

- This method sucks in accuracy (+/- 1 magnitude error in absolute magnitude ➔ 50% error in distance)

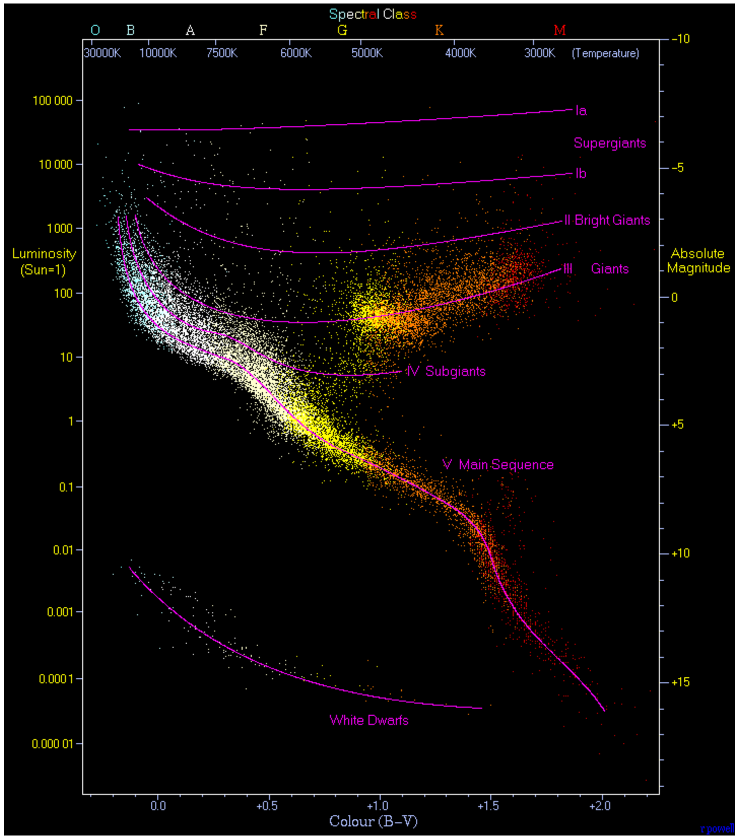
- But it can be applied to all stars

Determining distance: main sequence fitting

CANDLE

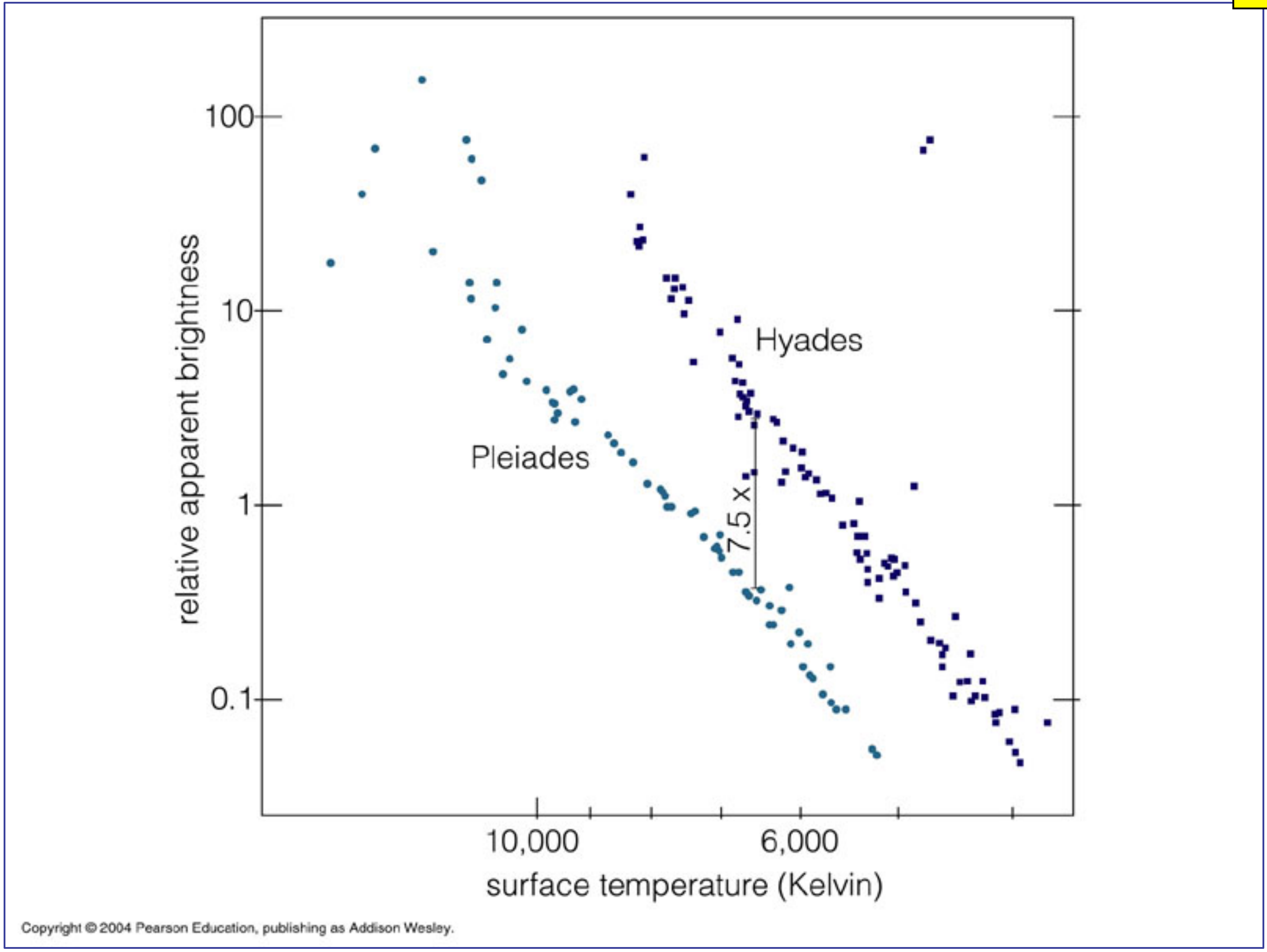
- Measure the colors and magnitudes of stars in a cluster (e.g., r and g-r)
- Plot the HR diagram: r vs. g-r
- Compare the the HR diagram of another cluster of known distance: M_r vs. g-r
- Find the vertical offset in HR diagram between the main sequences of the two clusters ➔ distance modulus

$$m - M = 5\log d + 5$$



Determining distance: main sequence fitting

CANDLE



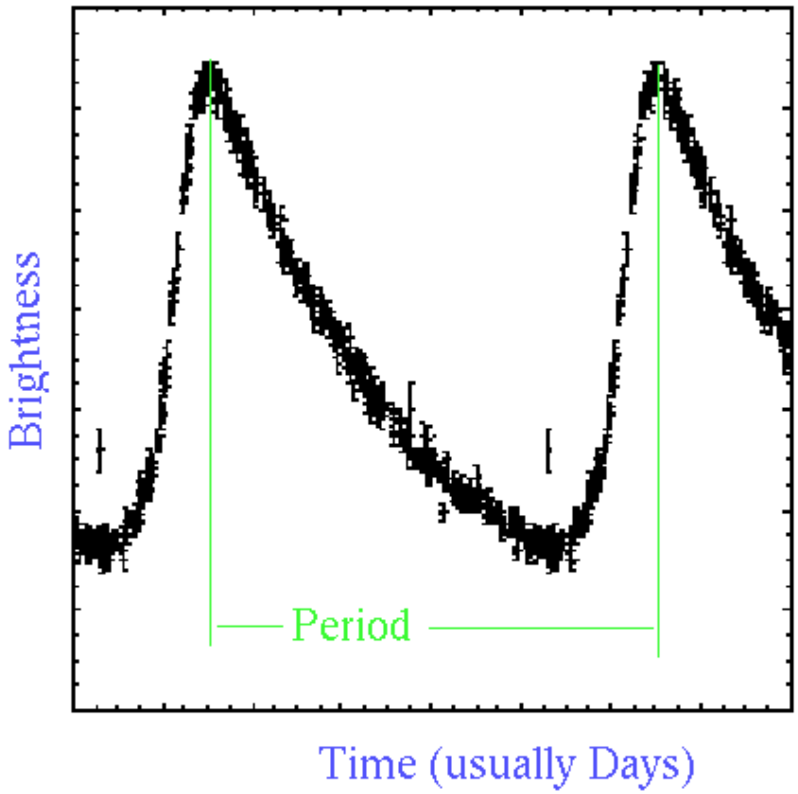
Determining distance: variable stars

Cepheid variables:
Pop I giants,   M ~ 5-20 M_{sun}

Pulsation due to feedback loop:

An increase in T
➔ HeIII (doubly ionized He)
➔ high opacity
➔ radiation can't escape
➔ even higher T and P
➔ atmosphere expands
➔ low T
➔ HeII (singly ionized He)
➔ low opacity
➔ atmosphere contracts
➔ rinse and repeat…

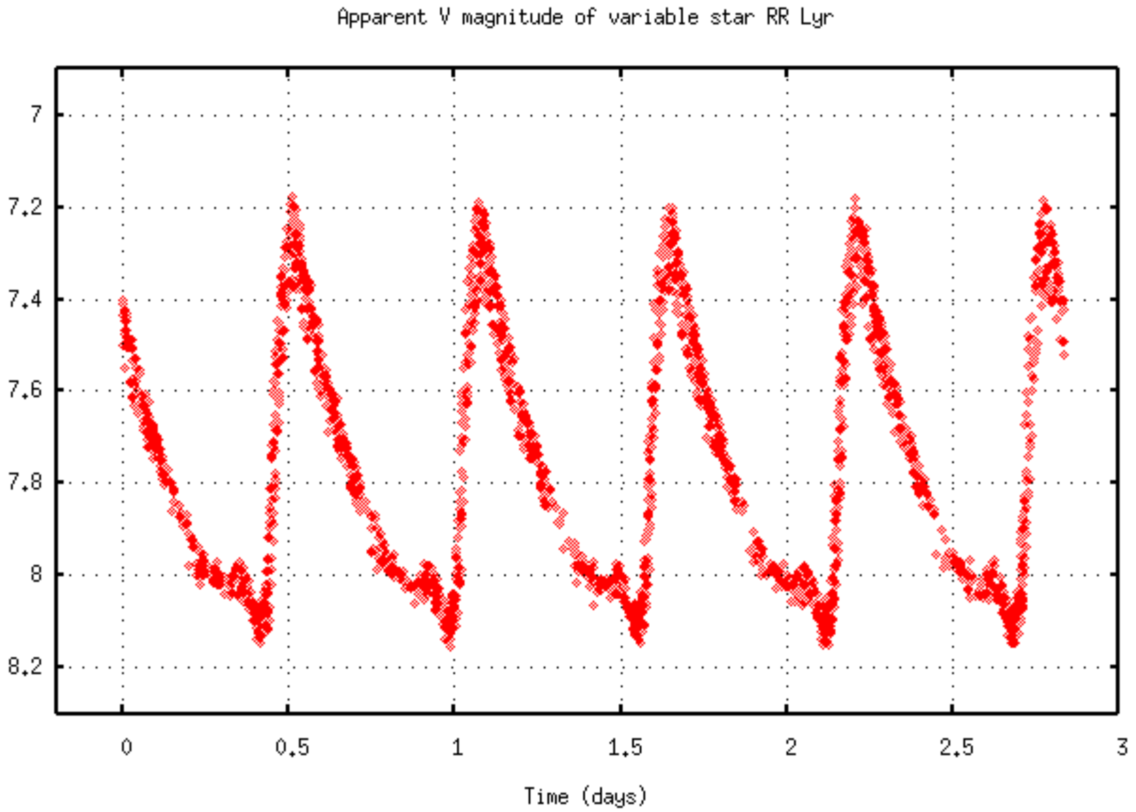
Data from a Well–Measured Cepheid



Determining distance: variable stars

<u>RR-Lyrae variables</u>:

Pop II dwarfs, M ~ 0.5 M_{sun}



Determining distance: variable stars

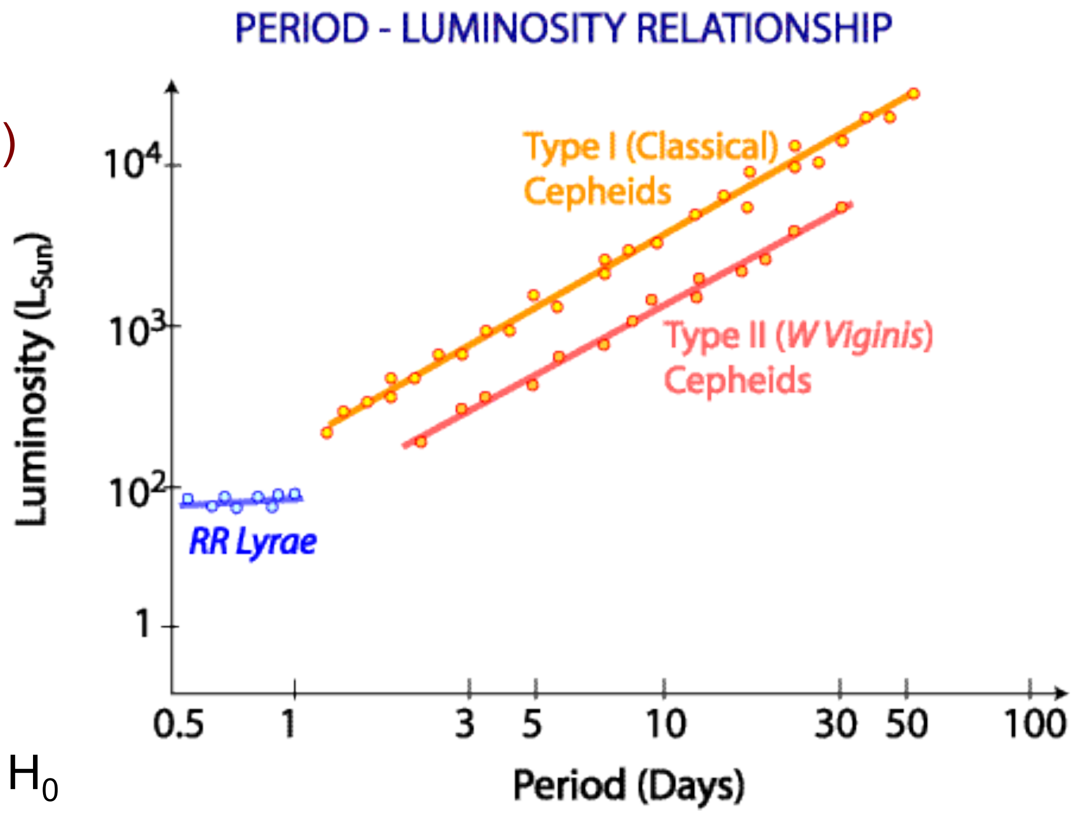
CANDLE

Variable stars have a tight period-luminosity relation

- Measure lightcurves: flux(t)
- Get period P
- From P-L relation, get L
- Use L to get distance

Very powerful method. Cepheids can be seen very far away. Used to measure H_0

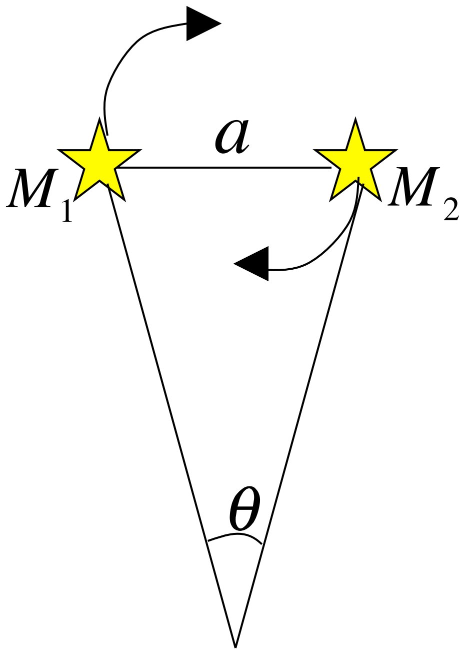
P-L relation is calibrated on local variables with parallax measurements



Determining distance: dynamical parallax

CANDLE

For binary star systems on main sequence



- Measure: period of orbit P, angular separation θ, fluxes f_1 and f_2
- Kepler's 3rd law: $P^2 = \frac{4\pi^2}{G(M_1 + M_2)} a^3$
- Assume $M_1 = M_2 = M_\odot$
- Use Kepler to get a
- Use θ to get preliminary distance
- Use distance and fluxes to get luminosities L_1, L_2
- Use mass-luminosity relation for main sequence to get better masses
- Iterate until convergence

Determining Luminosity

1. Measure flux or magnitude

   Measure distance

→

$$L = 4\pi d^2 f$$

$$M = m - 5\log d - 5$$

2. Measure spectrum

   line ratios and widths (I.e., compare the star to a star of identical spectrum that has a known L)

→ T, L

Determining Temperature

1. Spectral lines ---> spectral type

2. Colors
   (cheap and accurate)

3. Blackbody fitting or Wien's law: hν ~ 2.8kT
   (not good for very hot or cool stars)

4. Stellar atmosphere modeling
   (uncertain)

5. Measure angular size and flux
   (need very high resolution imaging)

$$R = \theta_R d \qquad L = 4\pi d^2 f$$

$$L = 4\pi R^2 \sigma T^4 \rightarrow T = \left(\frac{L}{4\pi R^2 \sigma} \right)^{\frac{1}{4}} \rightarrow T = \left(\frac{4\pi d^2 f}{4\pi (\theta_R d)^2 \sigma} \right)^{\frac{1}{4}}$$

$$T = \left(\frac{f}{\sigma \theta_R^2} \right)^{\frac{1}{4}}$$

Determining Radius

- Very difficult because angular sizes are tiny.

  The sun at 1pc distance has an angular radius of ~0.5mas

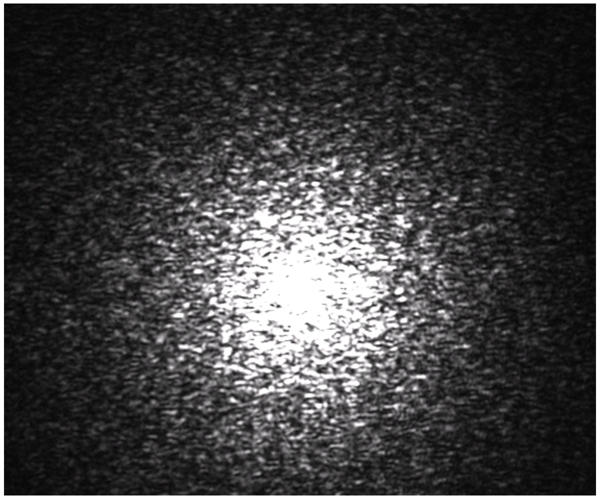
- Important because of:
  - surface gravity $g \sim \frac{GM}{R^2}$
  - density $\rho \sim \frac{M}{R^3}$
  - Temperature $T \sim \left(L/R^2\right)^{1/4}$
  - testing models

Determining Radius: Interferometry

Interferometry yields high resolution images

1. Speckle interferometry

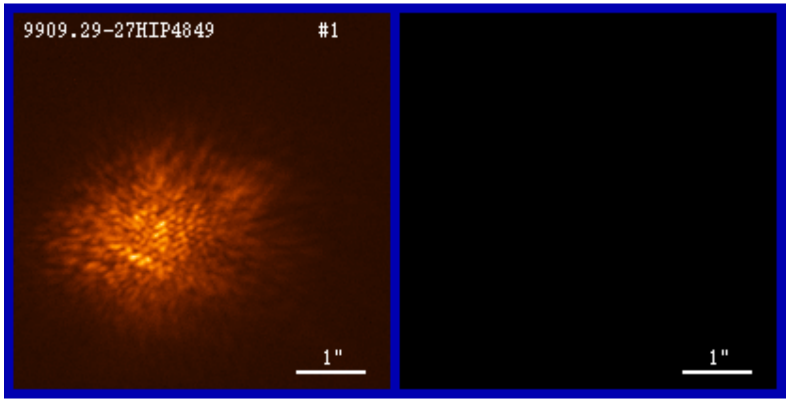
   ~ 0.02 as



Diffraction limited:

$$\theta = \frac{1.22\lambda}{D_T}$$

e.g., for HST (D_T=2.4m), θ =0.05 as



Determining Radius: Interferometry

2. Phase interferometry

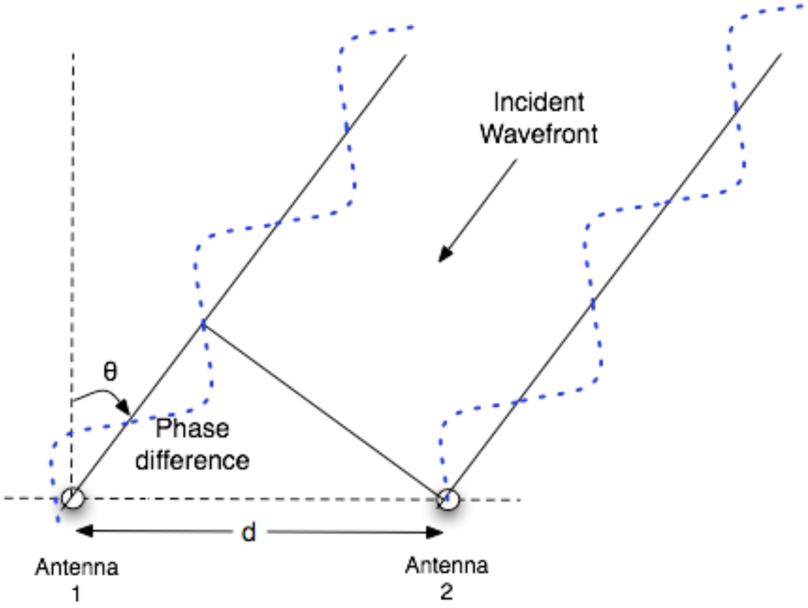
~ 0.01 as

Fringes disappear when

$$d \times \theta = \lambda/2$$

3. Intensity Interferometry

~ 0.5 mas



The sun’s radius could be measured out to ~1-10pc

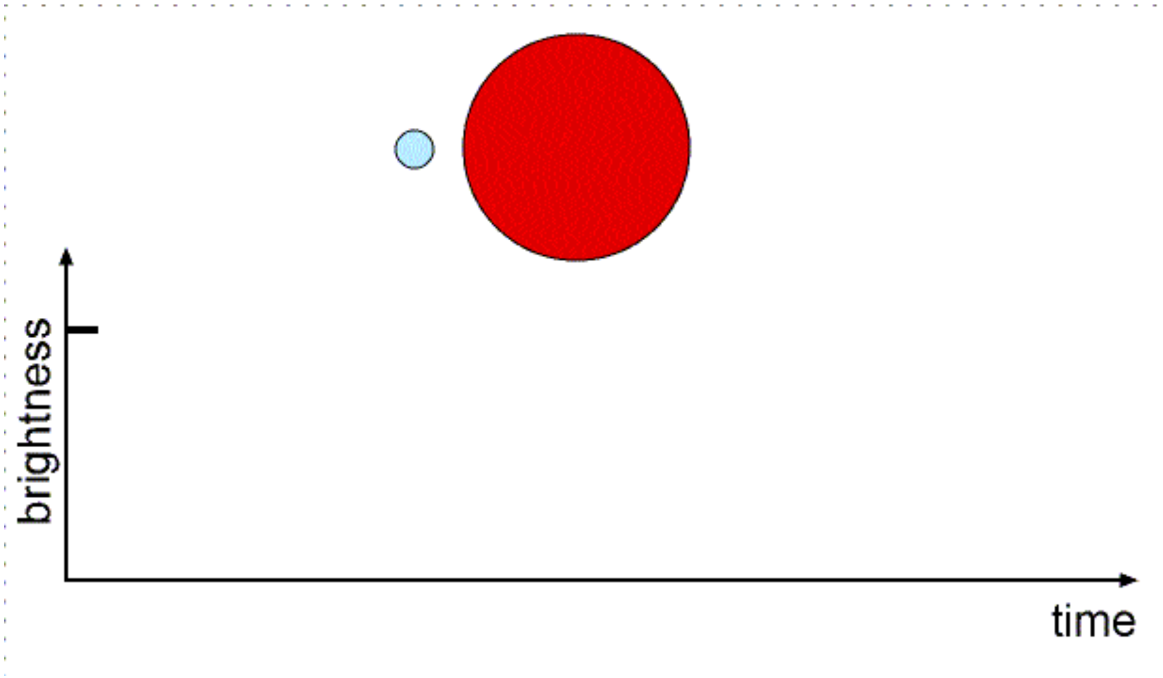
Determining Radius

- Luminosity + Temperature → Radius
- Baade-Wesselink (moving atmosphere)
- Lunar Occultation

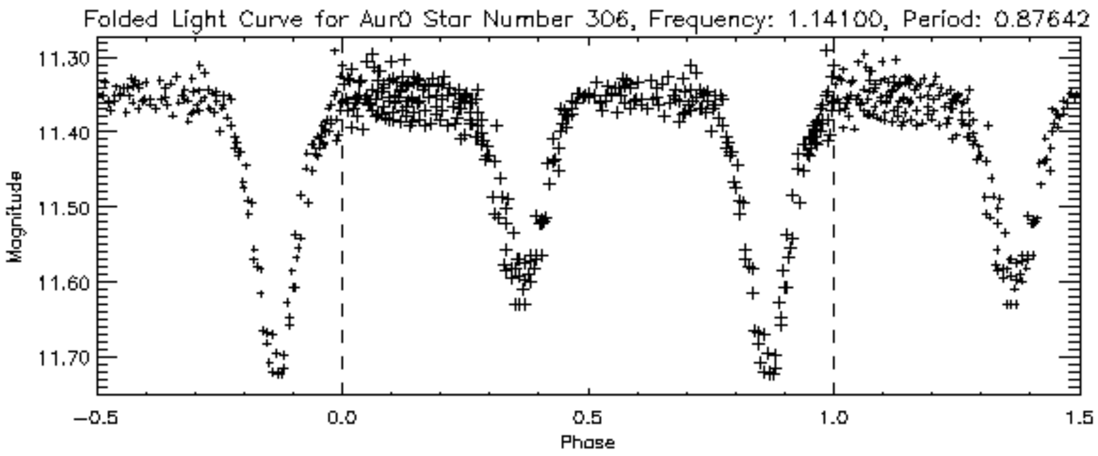
  As a star disappears behind the moon, a Fresnel diffraction pattern Is created that depends slightly on θ_R

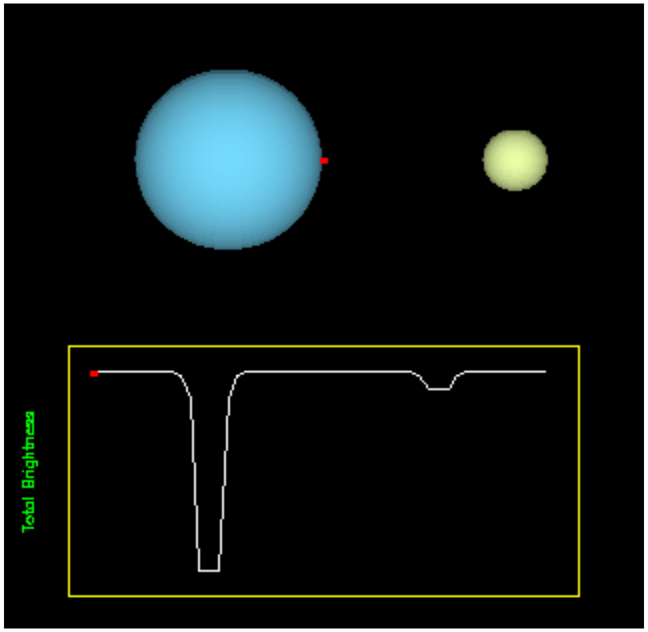
  ~ 2 mas, only good for stars in ecliptic.

Determining radius: eclipsing binaries



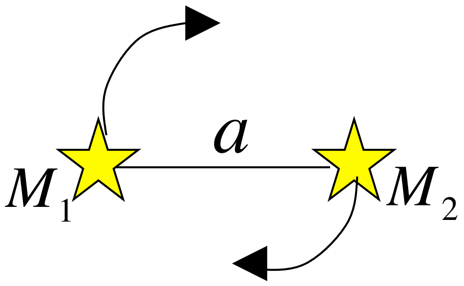
Determining radius: eclipsing binaries





Determining Mass

Only possible for binaries.



`

Kepler's 3rd law: $G(M_1+M_2)P^2 = 4\pi^2 a^3$

For Sun-Earth system:

$M_1 + M_2 \approx M_\odot$

$P = 1\text{yr}$

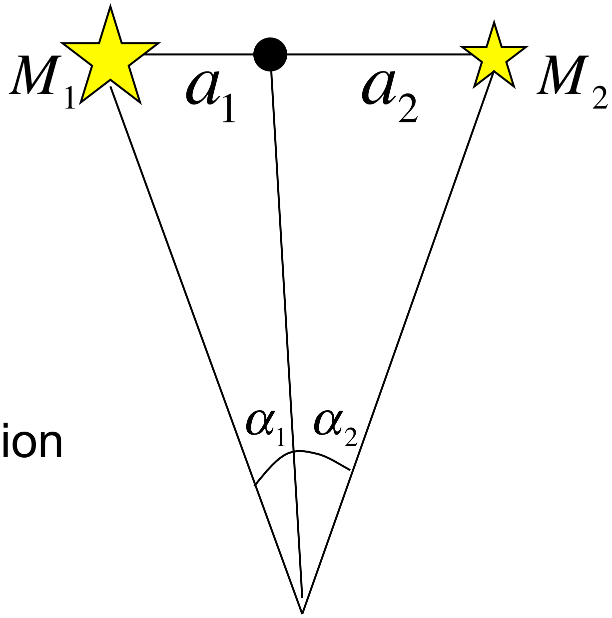
a=1AU

$\longrightarrow \quad GM_\odot (1\text{yr})^2 = 4\pi^2 (1\text{AU})^3$

$$\boxed{\left(\frac{M_1+M_2}{M_\odot}\right)\left(\frac{P}{1\text{yr}}\right)^2 = \left(\frac{a}{1\text{AU}}\right)^3}$$

Determining Mass: visual binaries

Visual binaries are binaries where the angular separation is detectable and the orbit can be traced out.



- Find the center of mass of the system (c.o.m. must move with constant velocity)

$$M_1 a_1 = M_2 a_2 \rightarrow \boxed{\frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{\alpha_2}{\alpha_1}}$$

- Measure the period, semi-major angular separation and distance.

$$\alpha = \alpha_1 + \alpha_2$$

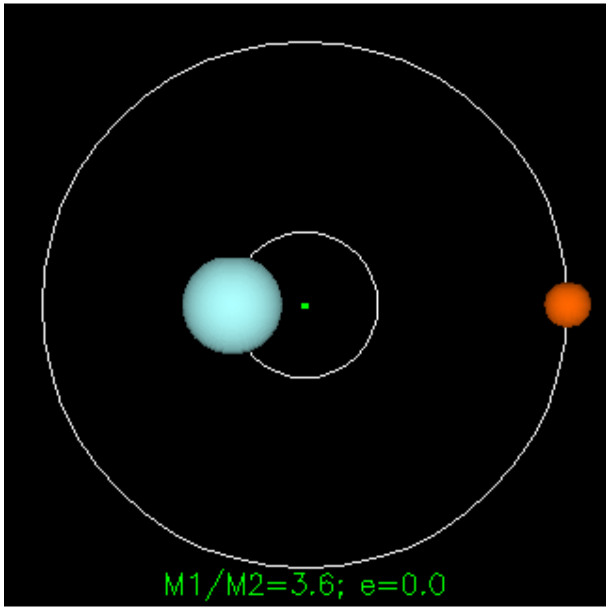
$$a = d\alpha$$

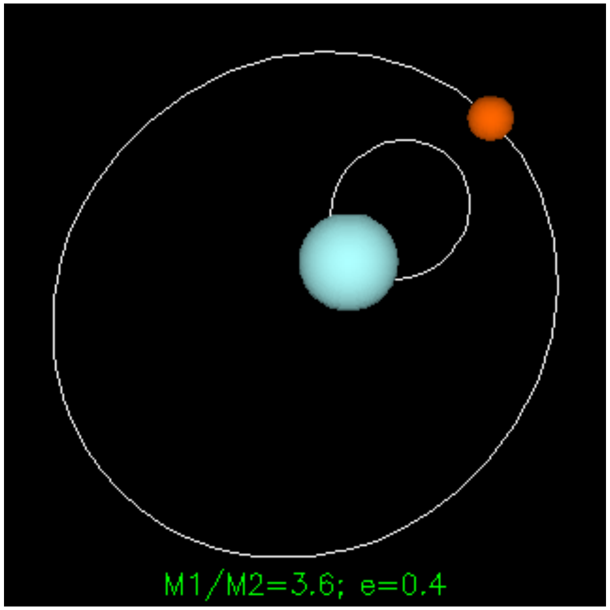
$$\boxed{M_1 + M_2 = \frac{(d\alpha)^3}{P^2}}$$

- Solve for individual masses. Very sensitive to distance errors: $M \sim d^3$

- If distance is not known, radial velocity data is sufficient to get a
  e.g., for a circular orbit: $v = 2\pi a / P$

Determining Mass: visual binaries





Determining Mass: visual binaries

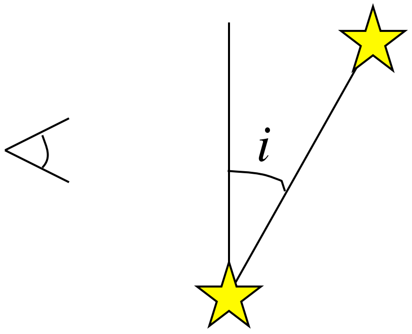
If the orbit is inclined, then we must know the inclination angle i

$$\frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{\alpha_2 \cos i}{\alpha_1 \cos i} = \frac{\alpha_2}{\alpha_1}$$

Don't need *i*

$$M_1 + M_2 = \left(\frac{\alpha}{\cos i}\right)^3 \frac{d^3}{P^2}$$

Need *i*



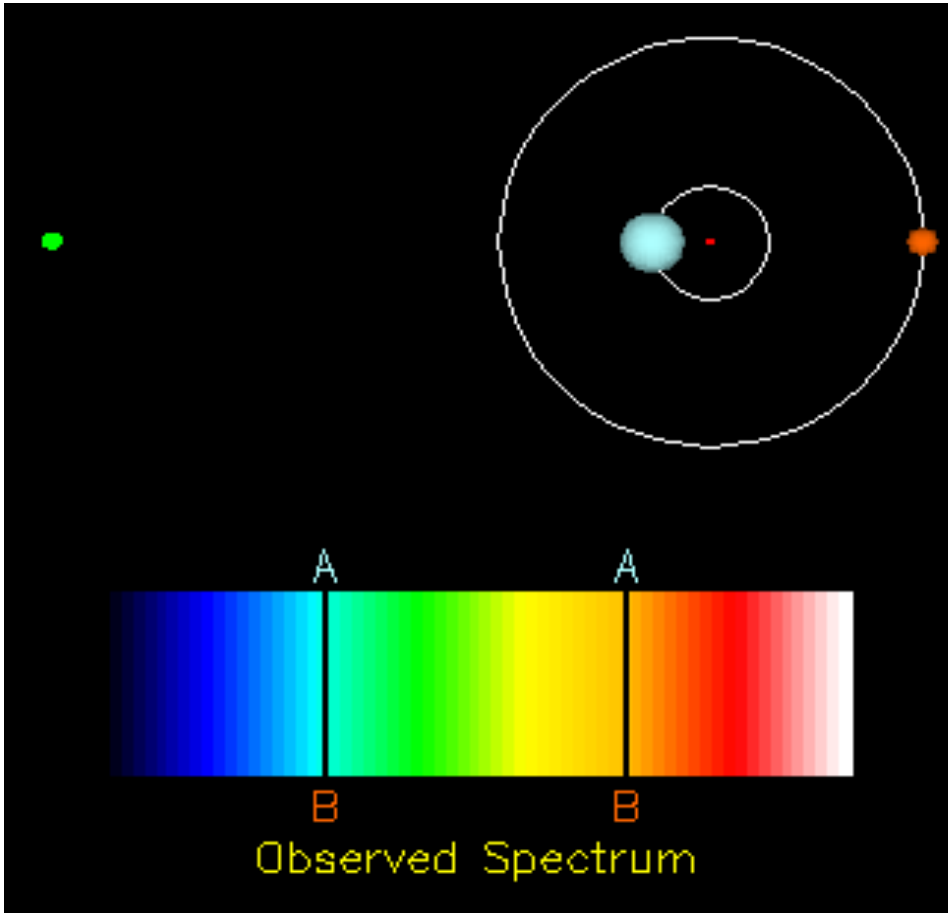
In an inclined orbit, the center of mass does not lie at the ellipse focus.
Find the projection that fixes this ➔ cos *i*

Determining Mass: spectroscopic binaries

Spectroscopic binaries are unresolved binaries that are only detected via doppler shifts in their spectrum.

- Single line binaries: single set of shifting spectral lines
- Double line binaries: two sets of spectral lines (one from each star)
- Two sets of lines, but for different spectral types

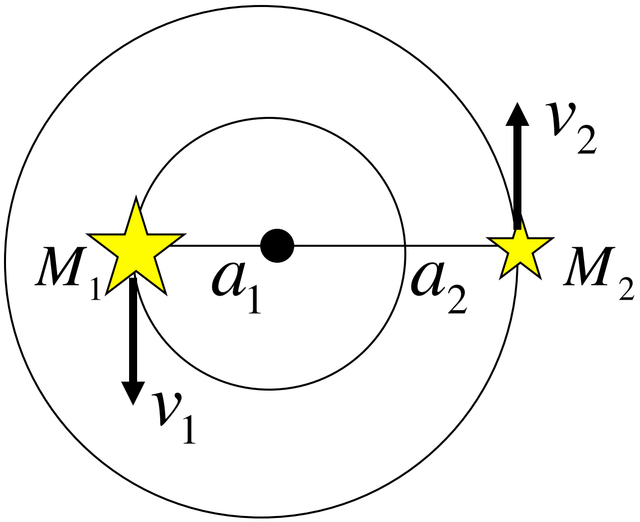
Determining mass: spectroscopic binaries



Determining Mass: spectroscopic binaries

- Measure velocities and period

$$\left.\begin{aligned} a_1 &= \frac{v_1 P}{2\pi} \\ a_2 &= \frac{v_2 P}{2\pi} \end{aligned}\right] \rightarrow \frac{a_1}{a_2} = \frac{v_1}{v_2} \rightarrow \frac{M_1}{M_2} = \frac{v_1}{v_2}$$



$$\left.\begin{aligned} a = a_1 + a_2 &= \frac{P}{2\pi}\left(v_1 + v_2\right) \\ M_1 + M_2 &= \frac{a^3}{P^2} = \frac{P\left(v_1 + v_2\right)^3}{\left(2\pi\right)^3} \\ v_{r,1} + v_{r,2} &= \left(v_1 + v_2\right)\sin i \end{aligned}\right] \rightarrow M_1 + M_2 = \frac{P\left(v_{r,1} + v_{r,2}\right)^3}{\left(2\pi \sin i\right)^3}$$

- Can get a lower limit on M_1+M_2 since sin *i* <1

Determining Mass: spectroscopic binaries

Need to measure both radial velocities - only possible with double line binaries.

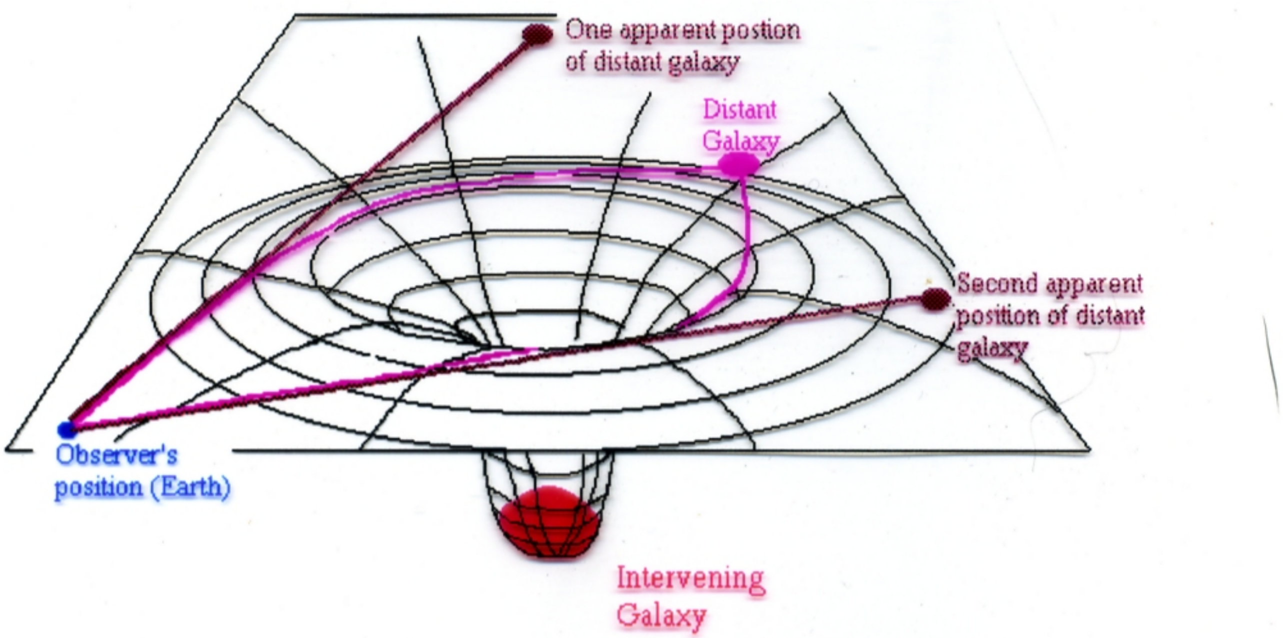
Need to know sin *i*. This is possible when:

- also an eclipsing binary: sin *i* = 1
- also a visual binary: measure orbit
- compute M_1+M_2 for a statistical sample where <$\sin^3 i$> is known
  (For an isotropic distribution, <$\sin^3 i$>=0.42. However, no Doppler shift will be observed if *i*=0, so there is a selection effect that favors high inclinations)

Determining Mass: surface gravity

- Pressure broadening of spectral lines ➔ surface gravity $g = \frac{GM}{R^2}$
- Measure radius R ➔ $\boxed{M = \frac{gR^2}{G}}$
- Sensitive to errors in R

Determining Mass: Gravitational lensing



Discovery of Widest-separation Quasar Gravitational Lens

Suprime-Cam (g',r',i')
December 17, 2003

Subaru Telescope, National Astronomical Observatory of Japan

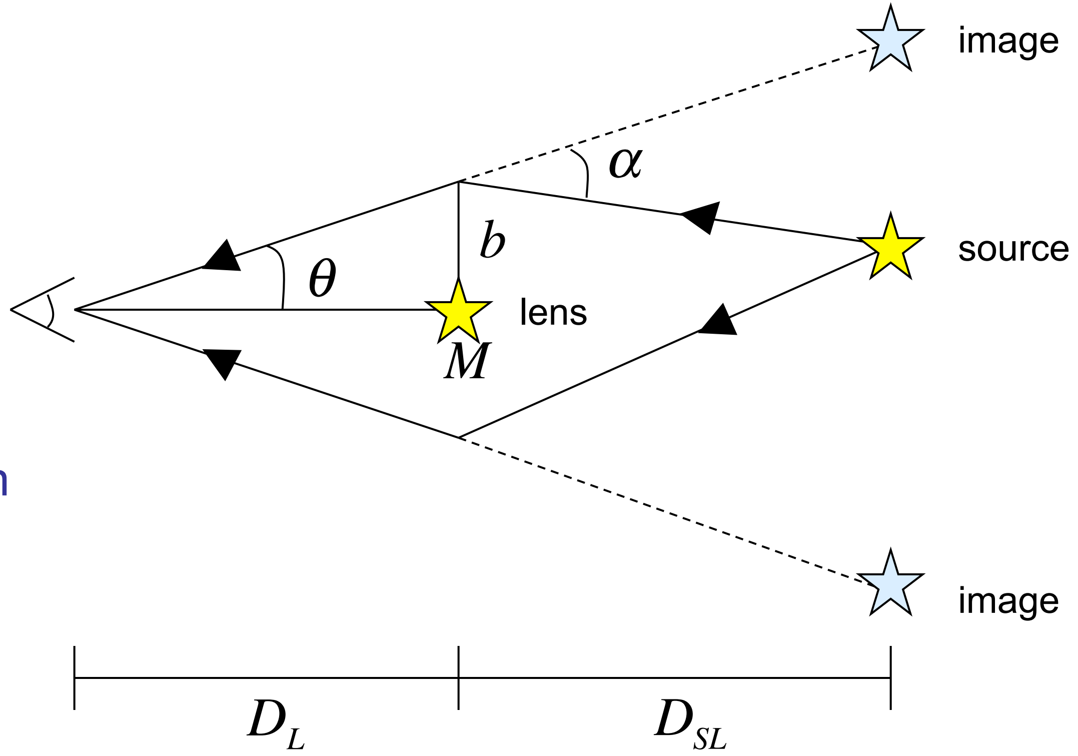
Determining Mass: Gravitational lensing

Determining Mass: gravitational microlensing

Case of lensing when multiple images are unresolved and we only detect an increase in the flux of one image.

General Relativity:

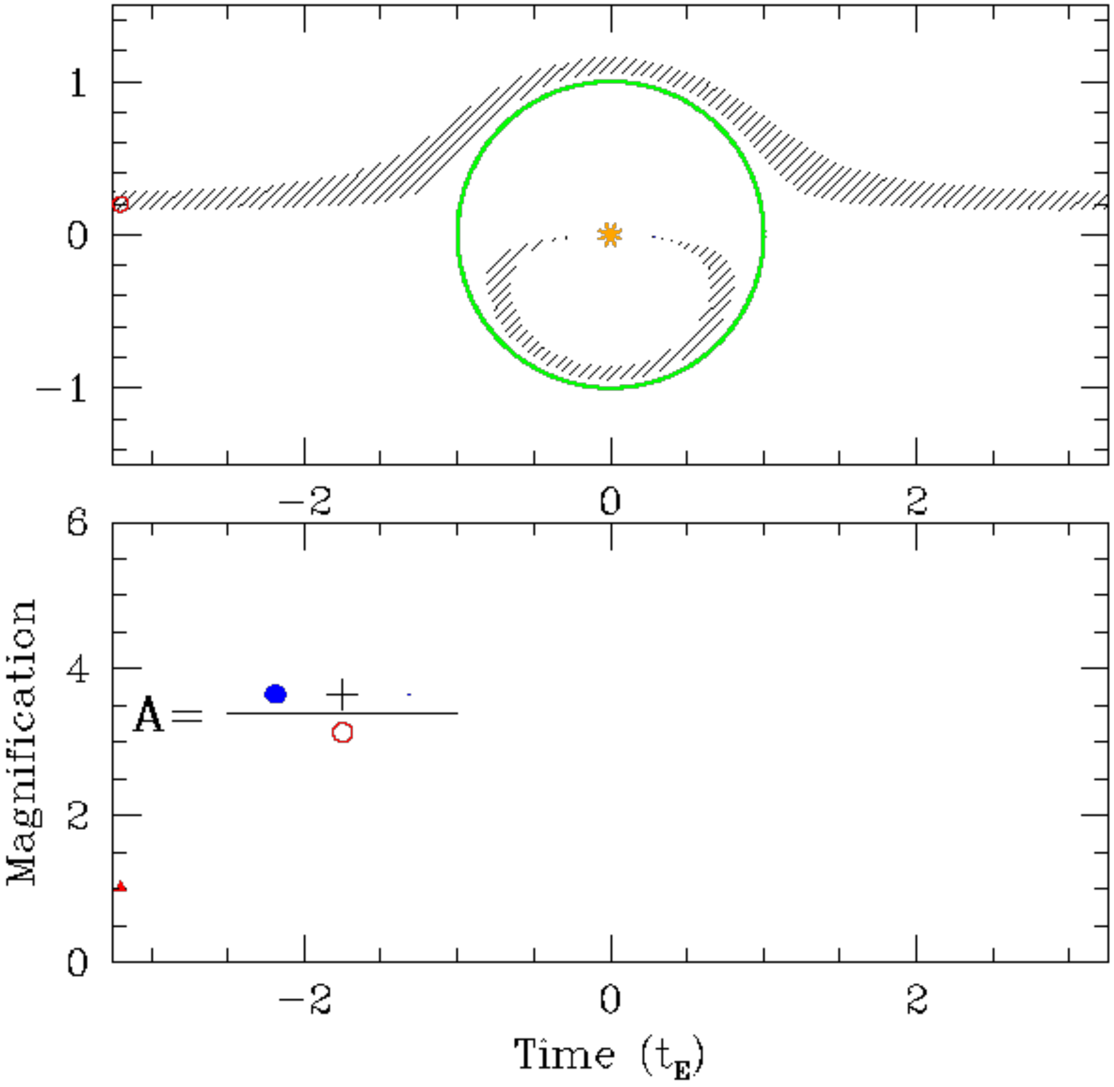
$$\alpha = \frac{4GM}{c^2 b}$$



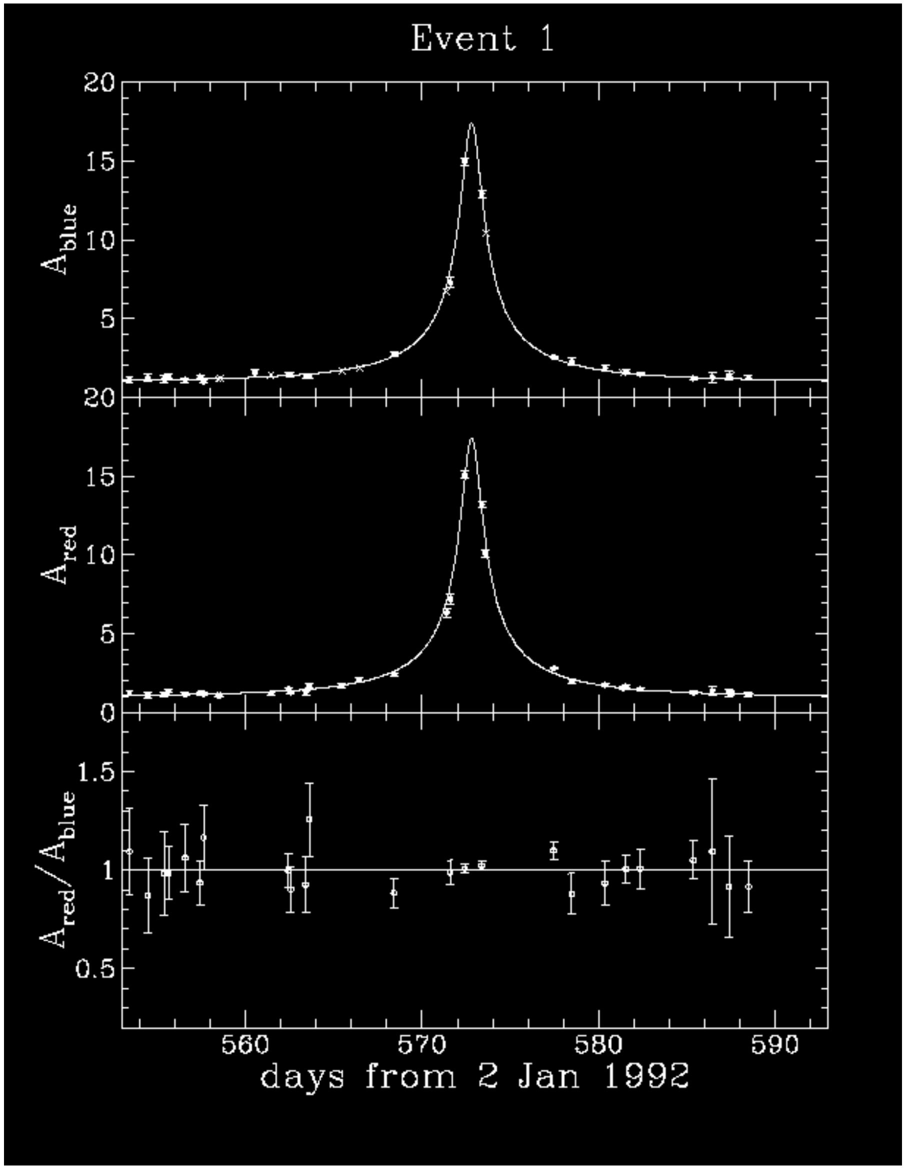
Can compute the amplification of light due to multiple images

$$A = f\left(M, D_L, D_{SL}, \theta\right)$$

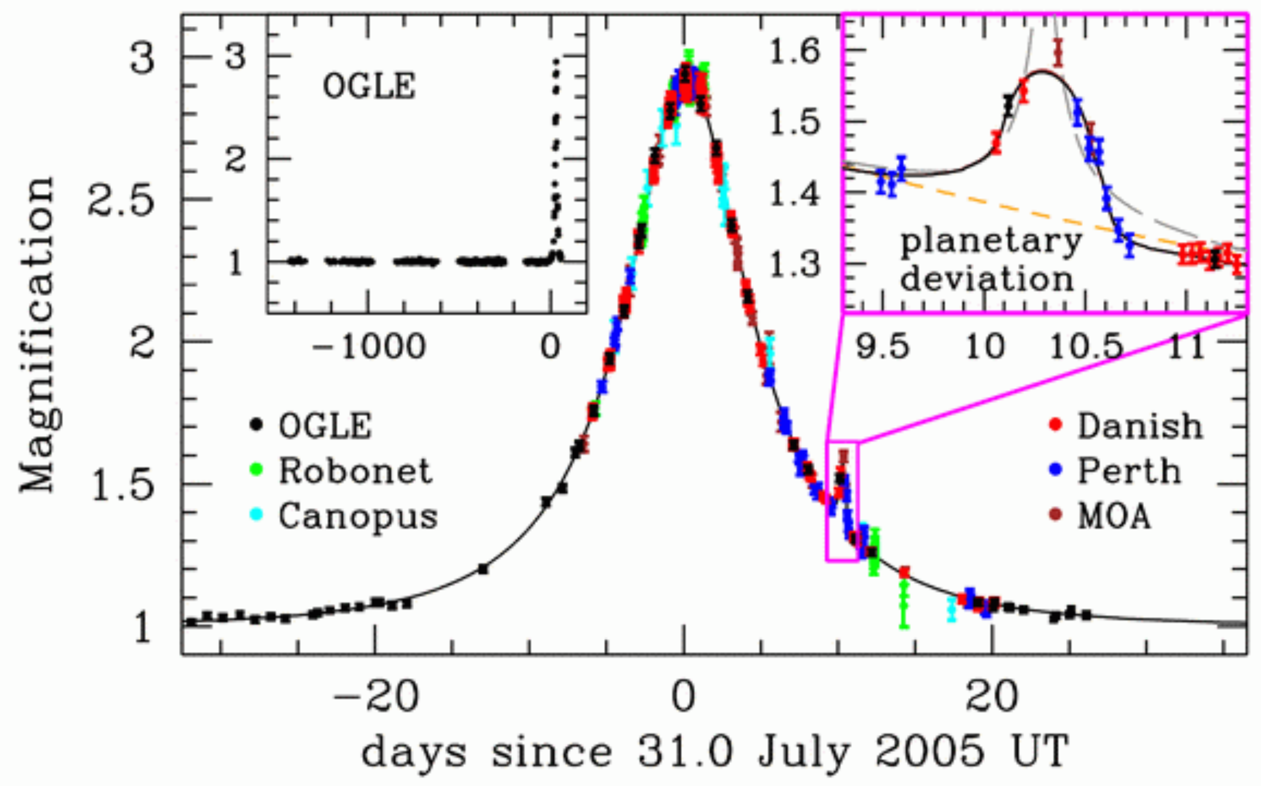
Determining mass: gravitational microlensing



Determining mass: gravitational microlensing



Determining mass: gravitational microlensing



Determining Chemical Composition

Theoretical: fractional abundances by mass

X: Hydrogen Y: Helium Z: metals Sun: (0.7, 0.28, 0.02)

Observational: number density relative to Hydrogen, normalized to sun.

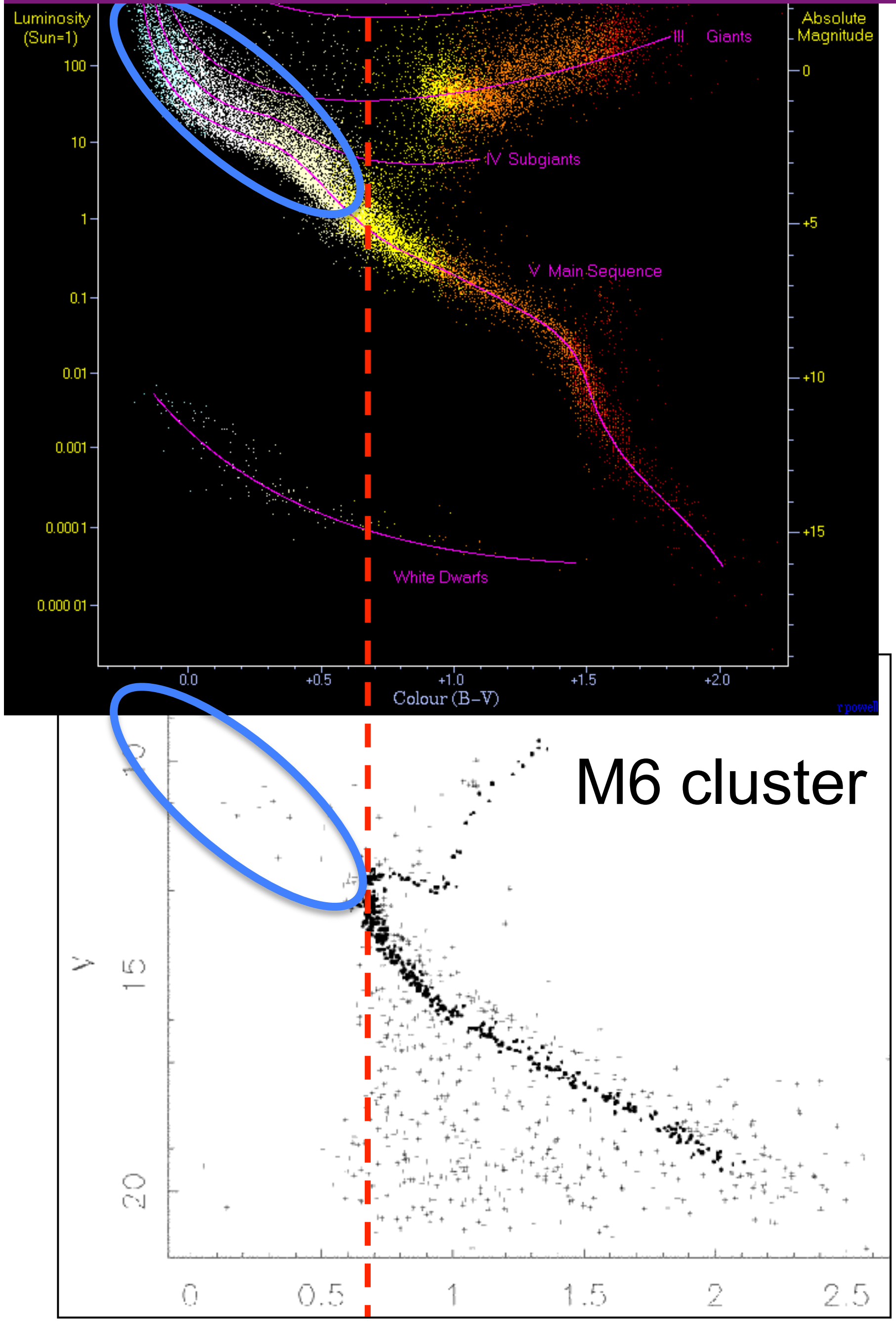
e.g., iron abundance $$\left[\mathrm{Fe/H}\right] \equiv \log\left(\frac{n(\mathrm{Fe})}{n(\mathrm{H})}\right)_* - \log\left(\frac{n(\mathrm{Fe})}{n(\mathrm{H})}\right)_\odot$$

So, for solar abundance: $\left[\mathrm{Fe/H}\right] = 0$, 10 times more iron: $\left[\mathrm{Fe/H}\right] = +1$

In our galaxy, $\left[\mathrm{Fe/H}\right]$ ranges from -4.5 to +1.

Need high resolution spectra + stellar atmosphere models to get abundances via fitting.

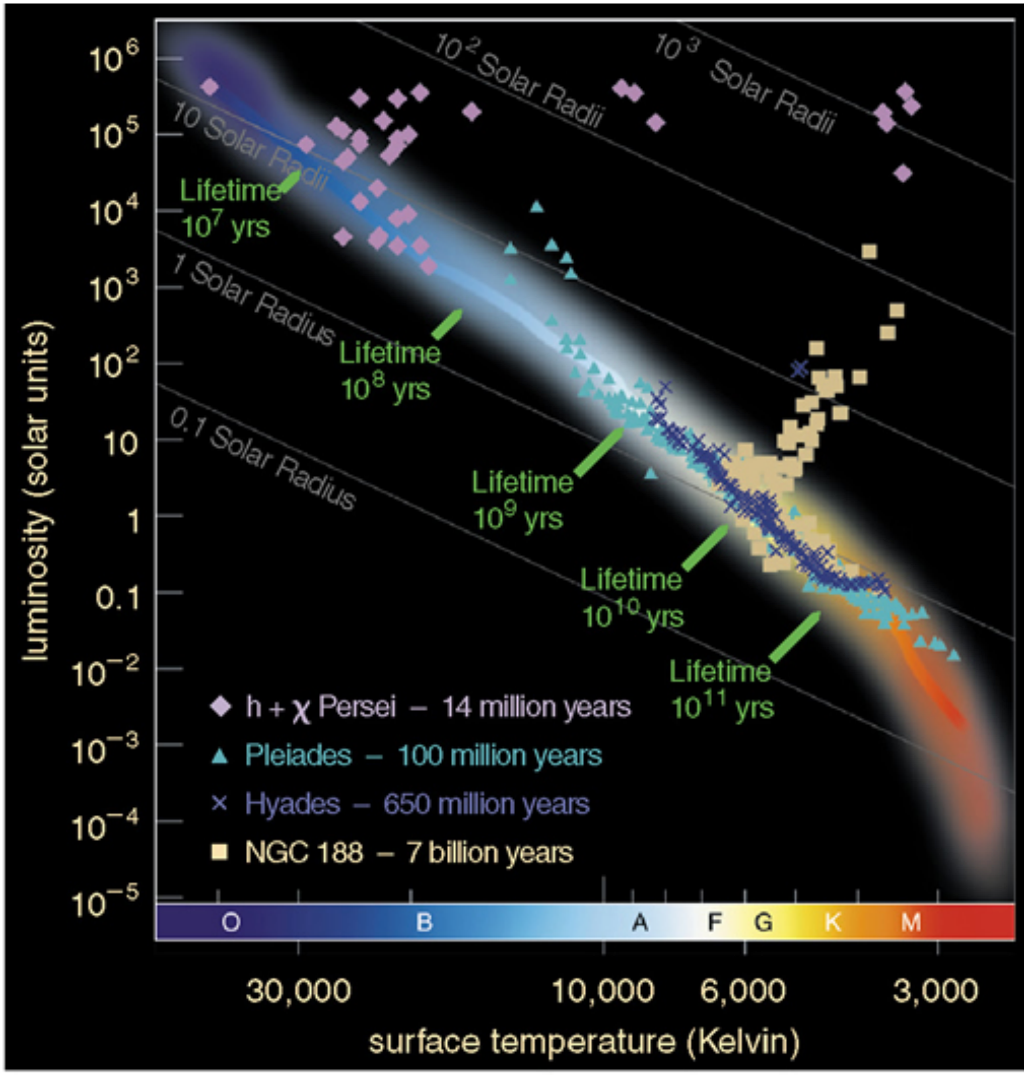
Determining age: Main Sequence turn-off



The lack of blue dwarfs tells us that the age of M6 is sufficiently old that the massive blue stars have died.

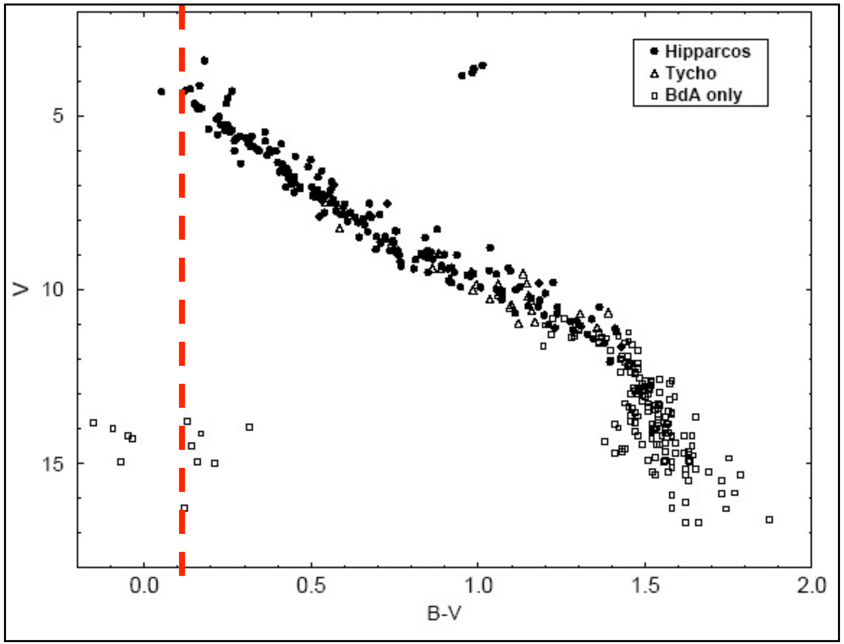
Yellow and red stars are still present because they live longer lives.

Determining age: Main Sequence turn-off

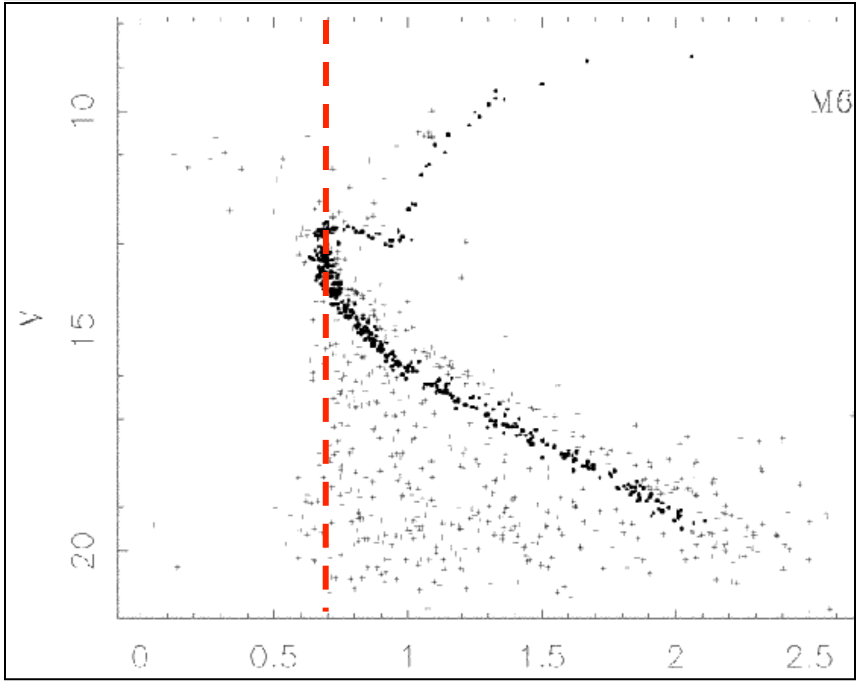


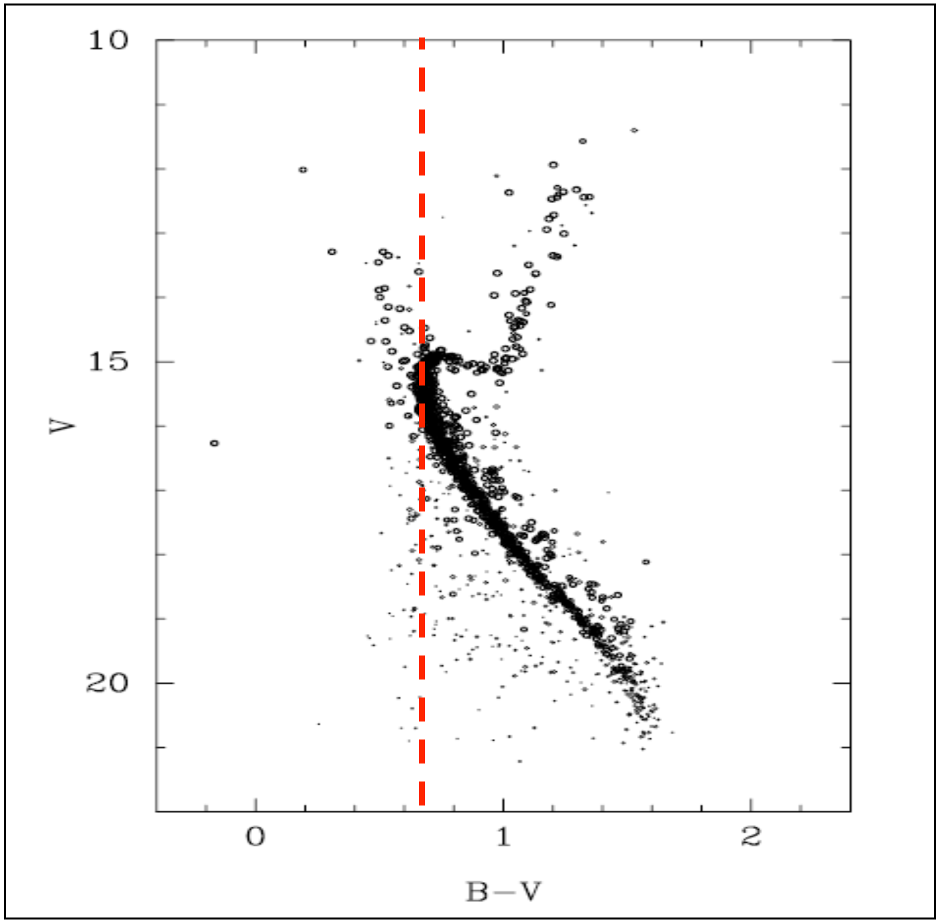
- All stars arrived on the main-sequence (MS) at about the same time.
- The massive stars on the top left of MS are the first to go.
- The cluster is as old as the most luminous star that remains on the MS.
- The position of the hottest, brightest star on a cluster's MS is called the *main-sequence turnoff point*.

Determining age: Main Sequence turn-off

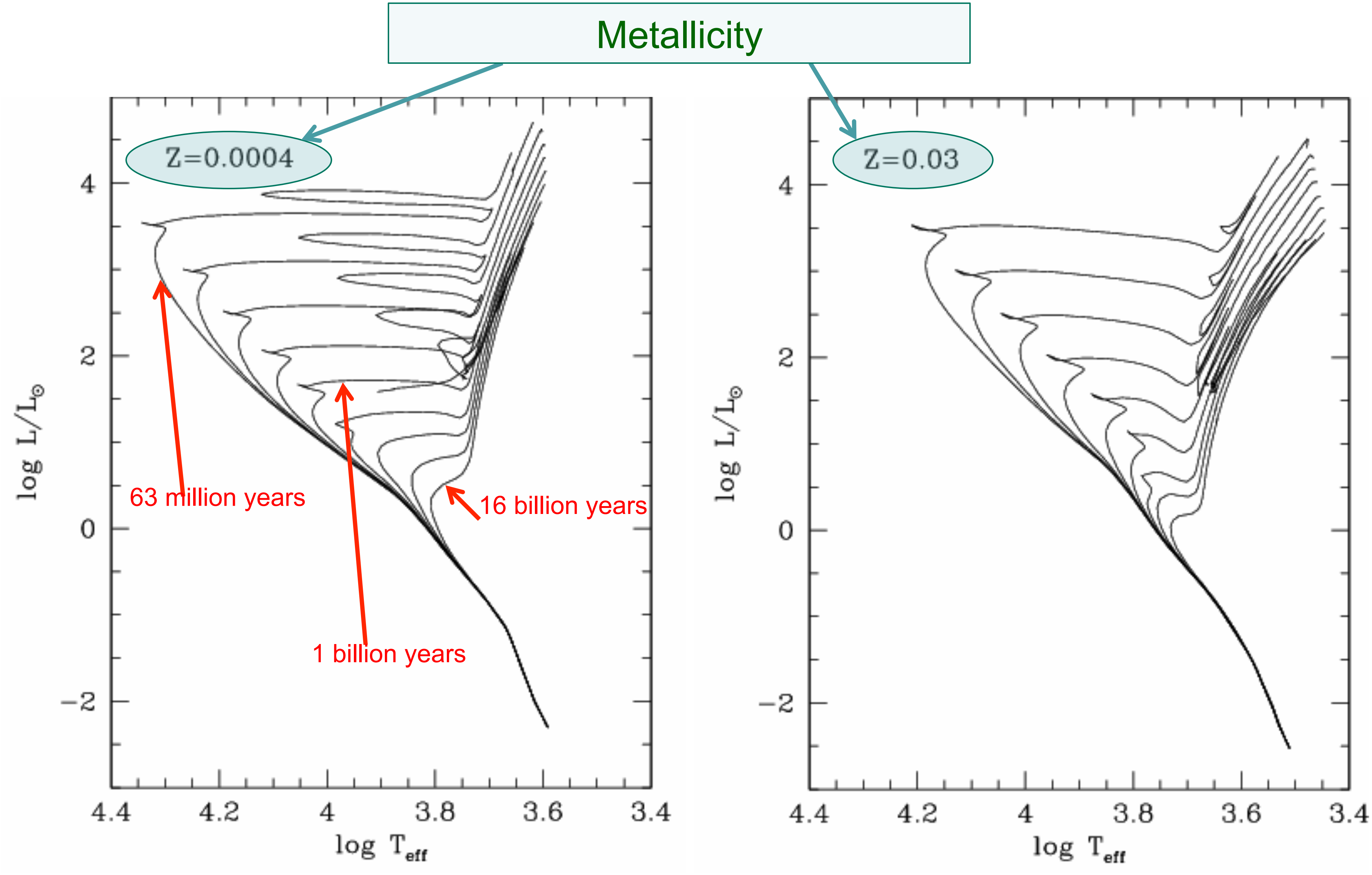


Which cluster is oldest?





Determining age: Isochrone Fitting



Metallicity

Z=0.0004

Z=0.03

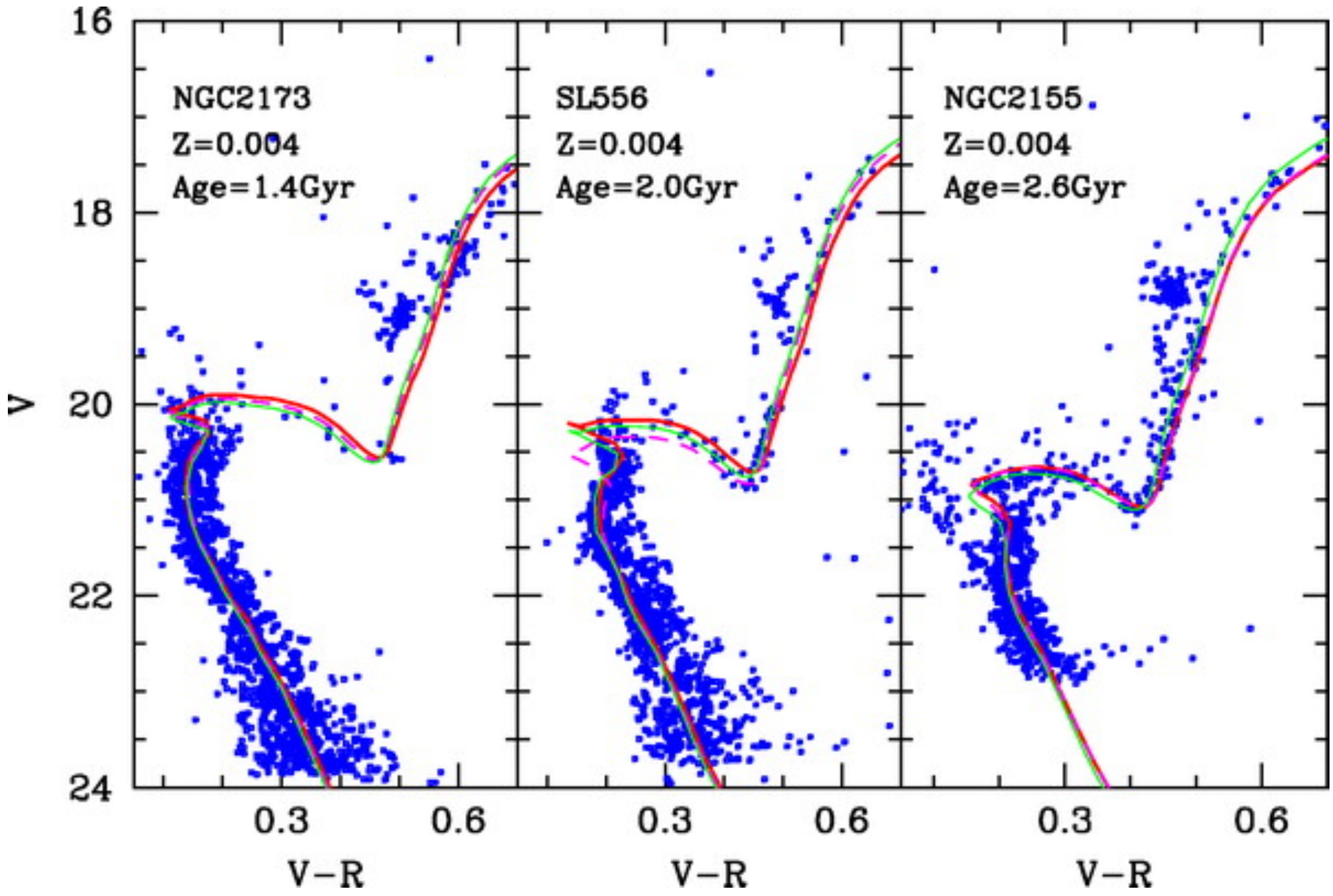
63 million years

1 billion years

16 billion years

Determining age: Isochrone Fitting

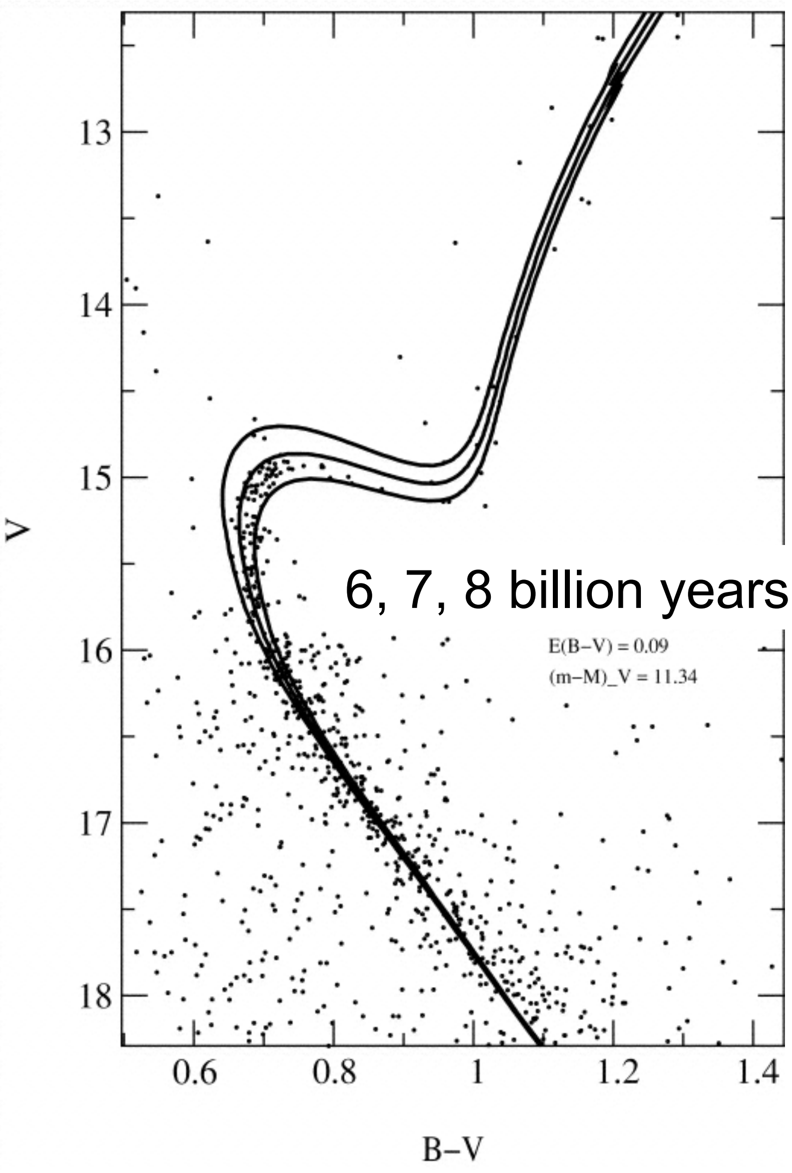
Open clusters



1.4 billion years 2 billion years 2.6 billion years

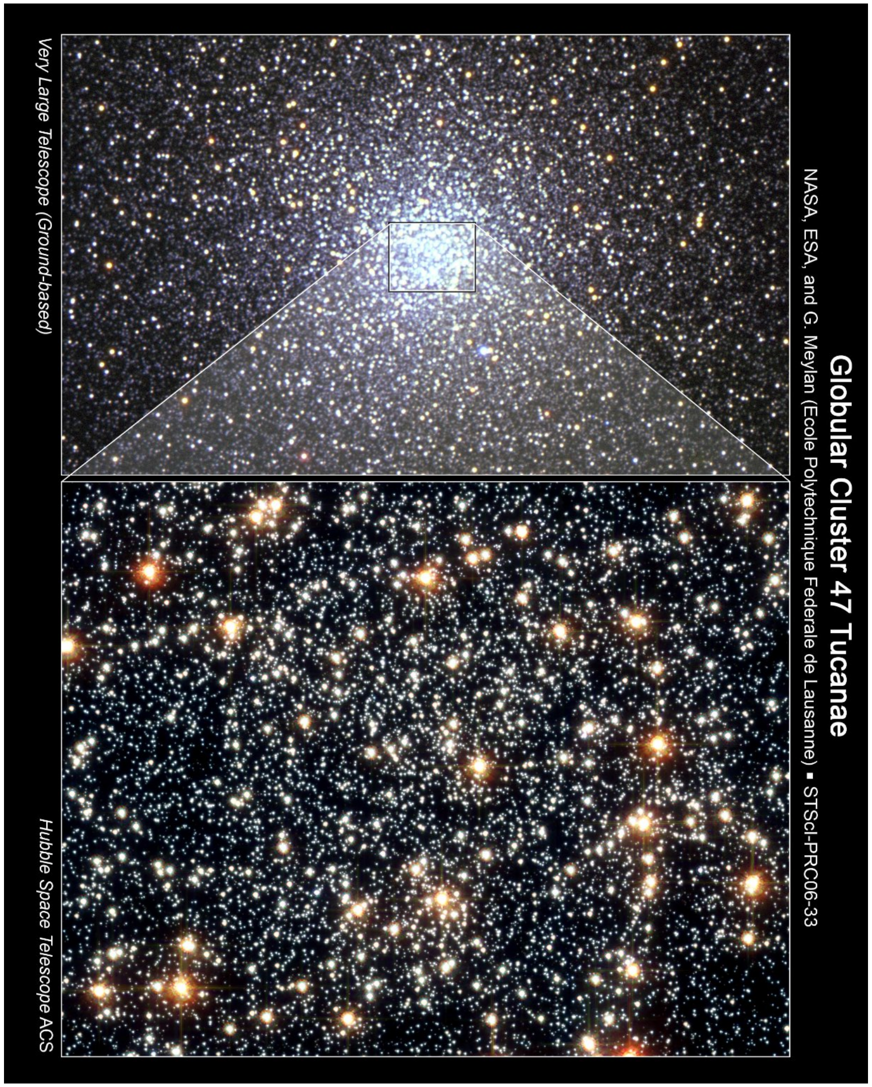
Determining age: Isochrone Fitting

NGC 188: 7 billion years

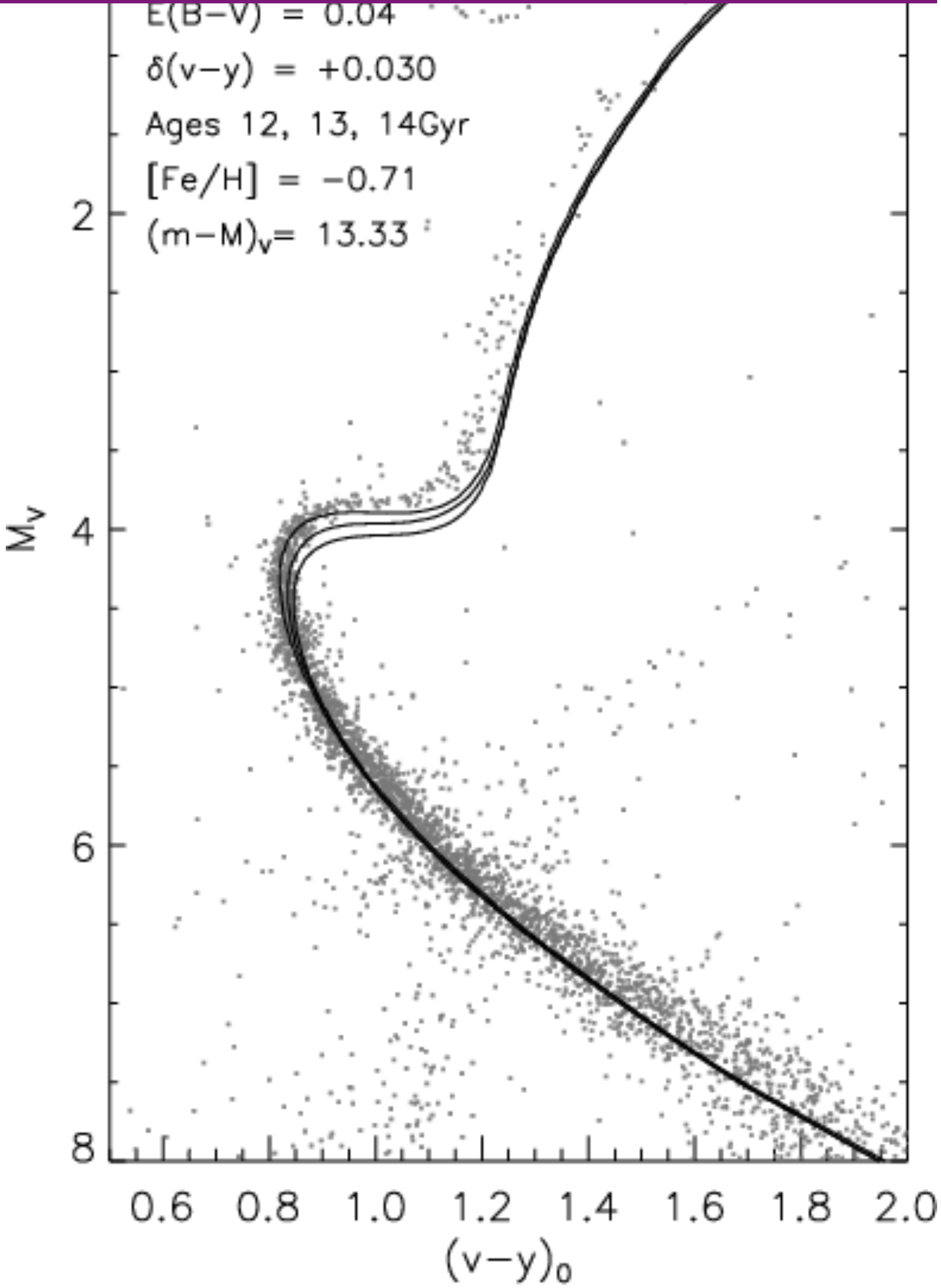


6, 7, 8 billion years

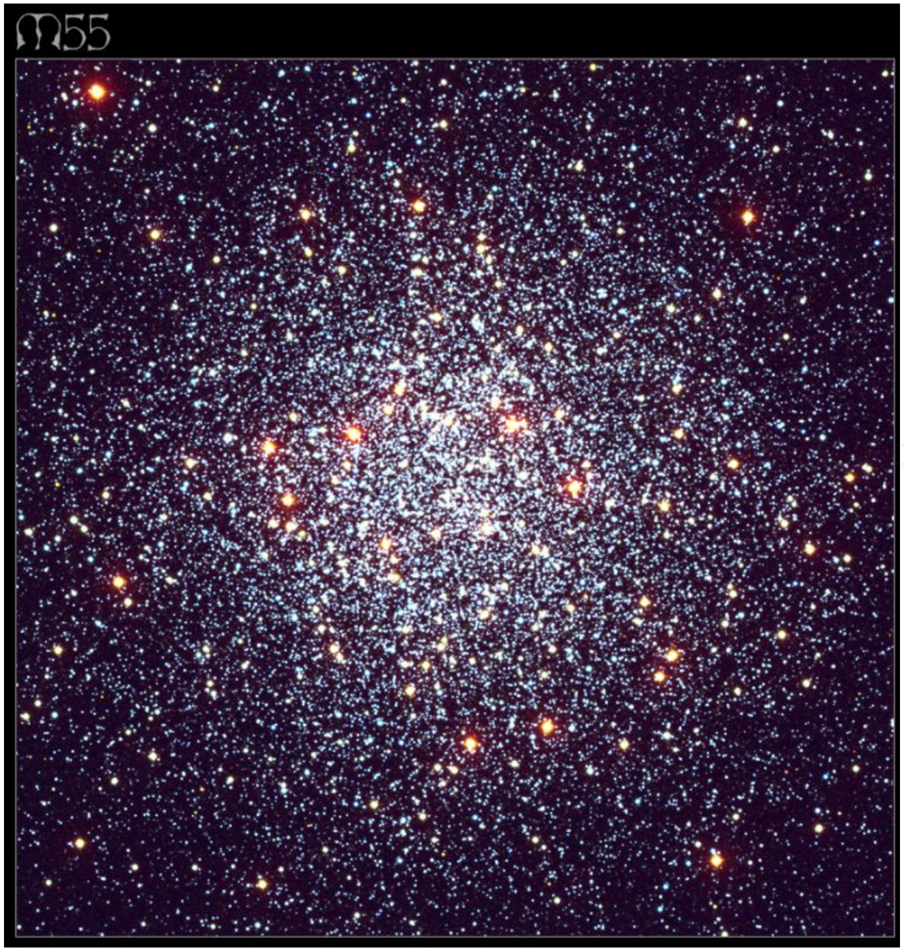
Determining age: Isochrone Fitting



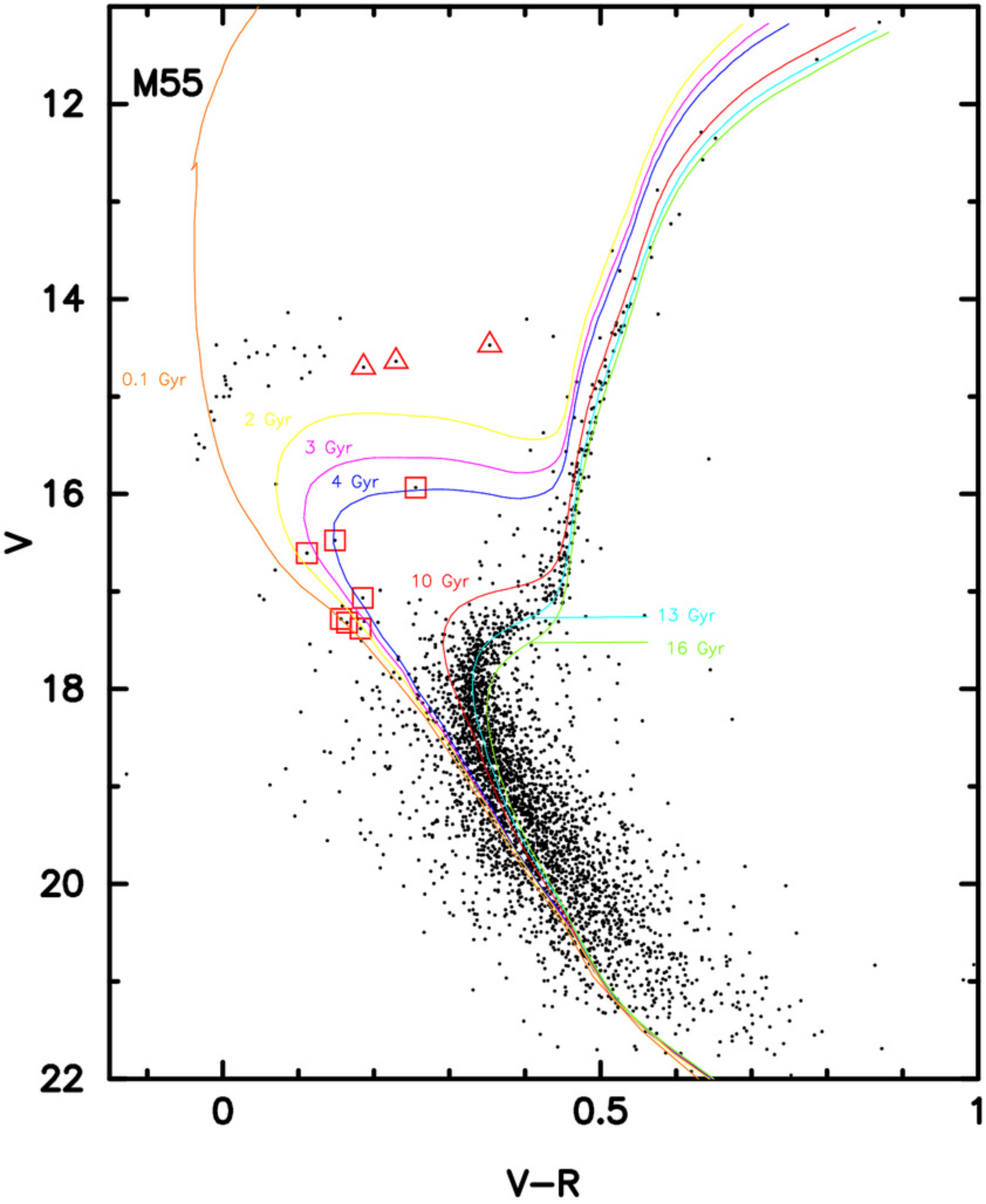
47 Tuc: 12 billion years



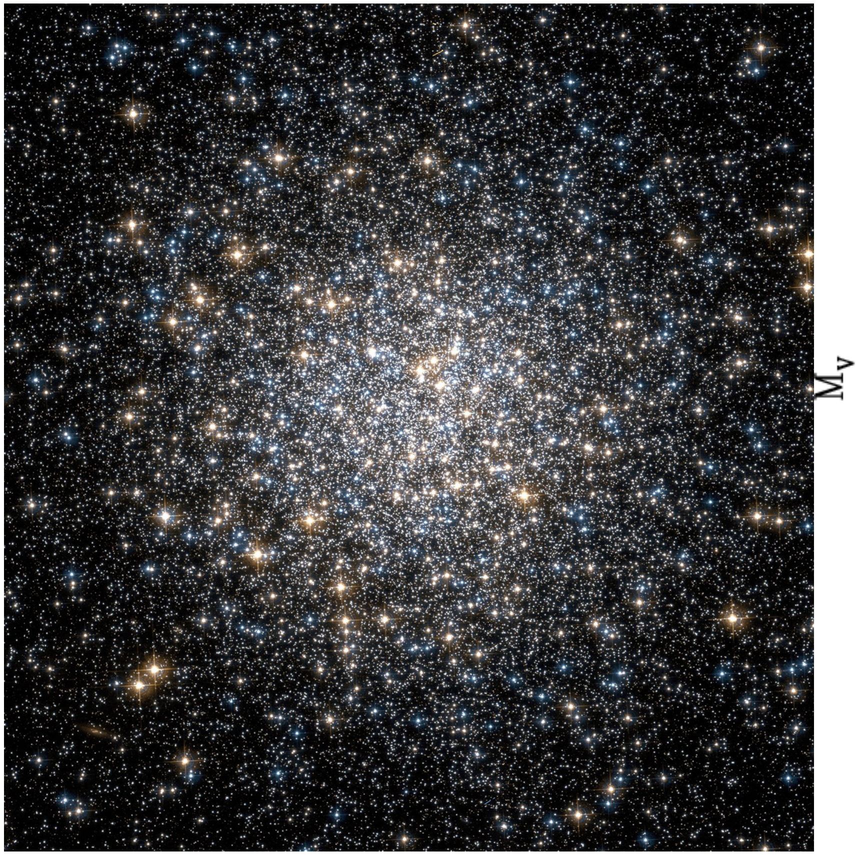
Determining age: Isochrone Fitting



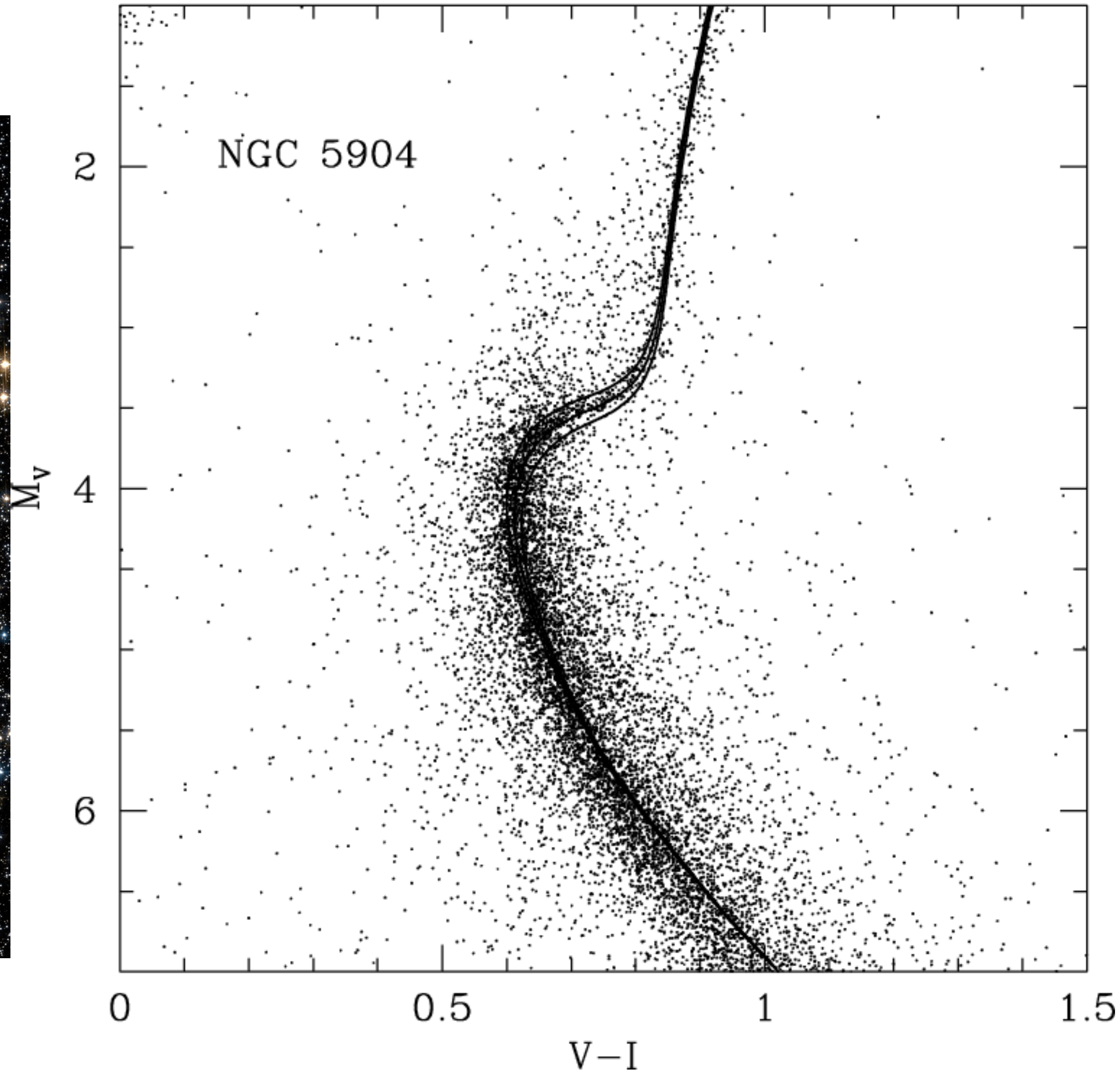
M55: 12.5 billion years



Determining age: Isochrone Fitting



M5: 13-14 billion years



Determining age: Isochrone Fitting

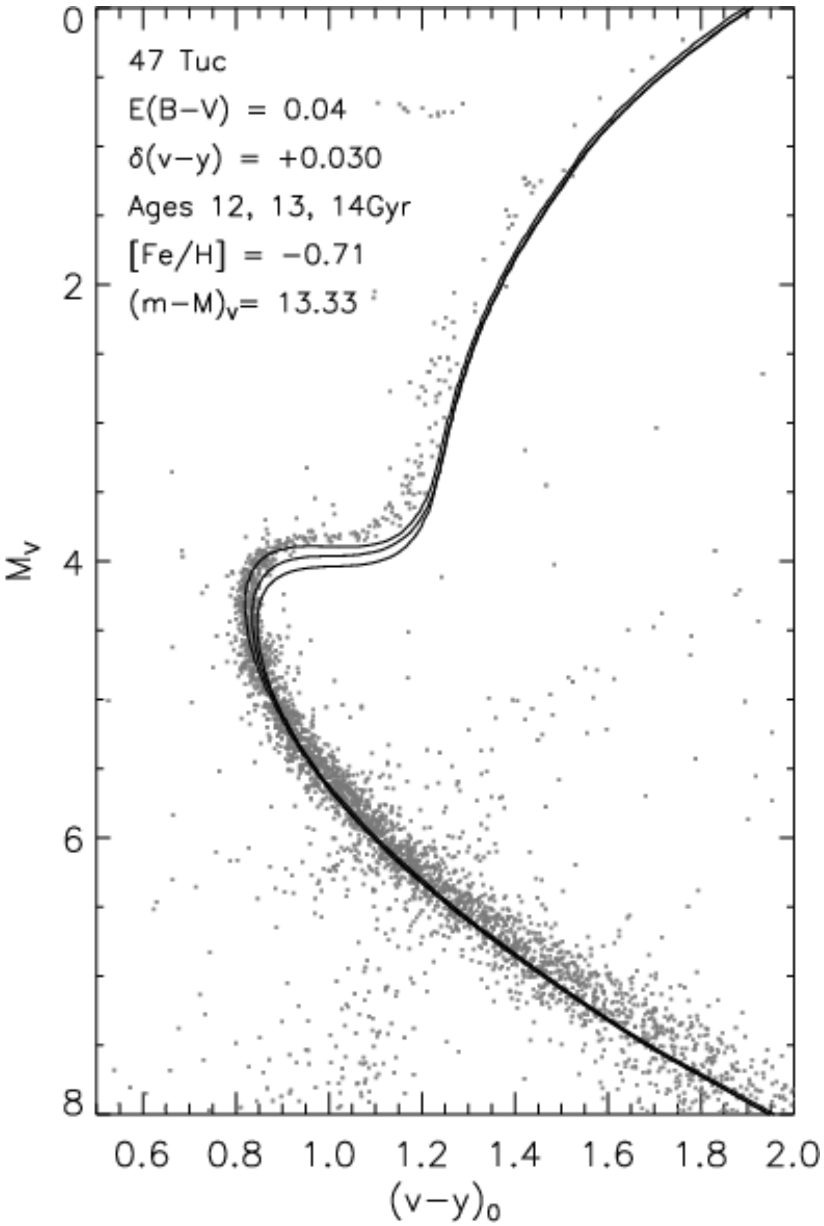
Measurement Errors

- errors in brightness/colors
- errors in distance/luminosity
- errors in cluster membership

Theoretical Errors

- uncertain chemical abundances
- uncertain amount of dust
- uncertain stellar physics

Typical error < 1 billion years



Global Energetics

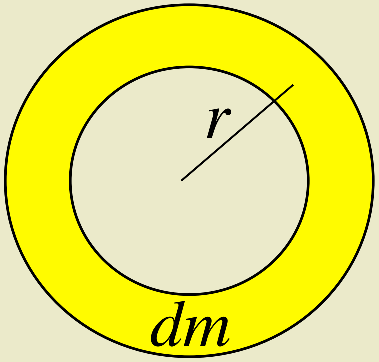
Total stellar energy

$$
\begin{aligned}
W &= \text{gravitational potential energy } \Omega \\
&\quad + \text{kinetic energy due to bulk motions (turbulence, pulsation)} \\
&\quad + \text{internal energy from microscopic processes } U
\end{aligned}
$$

Gravitational Potential Energy

Gravitational potential energy

Ω : energy required to assemble the star by collecting material from the outside universe.
= negative of energy required to disperse the star.



$$d\Omega = -\int_r^\infty \frac{Gmdm}{r'^2} dr' \quad = \quad -\frac{Gm}{r} dm$$

$$\Omega = -\int_0^M \frac{Gm}{r} dm$$

which depends on the density profile $\rho(r)$

Gravitational Potential Energy

e.g., for $\boxed{\rho(r) = \rho_0}$

$$\left.\begin{aligned} m &= \rho_0 \frac{4}{3}\pi r^3 \\ M &= \rho_0 \frac{4}{3}\pi R^3 \end{aligned}\right\} \rightarrow \frac{m}{M} = \left(\frac{r}{R}\right)^3 \rightarrow r = R\left(\frac{m}{M}\right)^{\frac{1}{3}}$$

$$\Omega = -\int_0^M \frac{Gm}{r} dm = -\frac{GM^{1/3}}{R}\int_0^M m^{2/3}\, dm \quad = -\frac{3}{5}\frac{GM^2}{R}$$

In general, $\boxed{\Omega = -q\frac{GM^2}{R}}$ and $q > \frac{3}{5}$ since $\rho \downarrow$ with r

$$\rho \sim r^{-1} \rightarrow q = \frac{2}{3} \qquad \rho \sim r^{-2} \rightarrow q = 1$$

Internal and Kinetic Energy

<u>Internal energy</u>

U : Kinetic energy due to motions of particles (thermal, not bulk)
+ energy in atomic and molecular bonds.

$$U = -\int_M E dm$$

Where E : local specific internal energy (erg/g)

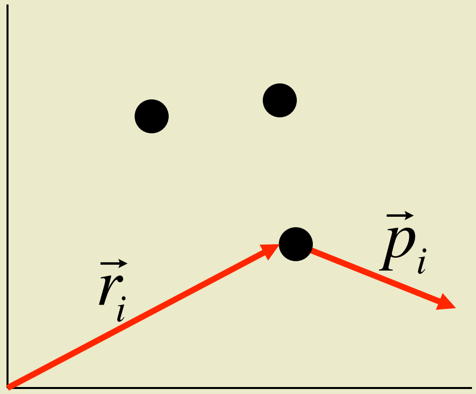
<u>Kinetic energy</u>

K : Total kinetic energy (thermal + bulk)

The Virial Theorem

Consider all particles in a star at positions $\vec{r}_i$ with momenta $\vec{p}_i$

$$Q = \sum_i \vec{p}_i \cdot \vec{r}_i$$



evaluate $\dfrac{dQ}{dt}$

what are the units of $\dfrac{dQ}{dt}$? (energy units: $\dfrac{m \cdot v \cdot r}{t} = m \cdot v^2$)

The Virial Theorem

$$\boxed{Q = \sum_i \vec{p}_i \cdot \vec{r}_i}$$

$$\begin{aligned}\frac{dQ}{dt} &= \frac{d}{dt}\sum_i m \cdot \dot{\vec{r}}_i \cdot \vec{r}_i \\ &= \frac{d}{dt}\sum_i \frac{d}{dt}\left(\frac{1}{2} m \cdot \vec{r}_i^{\,2}\right) \\ &= \frac{1}{2}\sum_i \frac{d^2}{dt^2}\left(m \cdot \vec{r}_i^{\,2}\right)\end{aligned}$$

$$\sum_i m \cdot \vec{r}_i^{\,2}$$

is just the total moment of inertia of the system.

$$\boxed{\frac{dQ}{dt} = \frac{1}{2}\frac{d^2 I}{dt^2}}$$

The Virial Theorem

$$\boxed{Q = \sum_i \vec{p}_i \cdot \vec{r}_i}$$

$$\frac{dQ}{dt} = \underbrace{\sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i}_{\text{A}} + \underbrace{\sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}}_{\text{B}}$$

B: $$\sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt} = \sum_i m \cdot \vec{v}_i \cdot \vec{v}_i$$

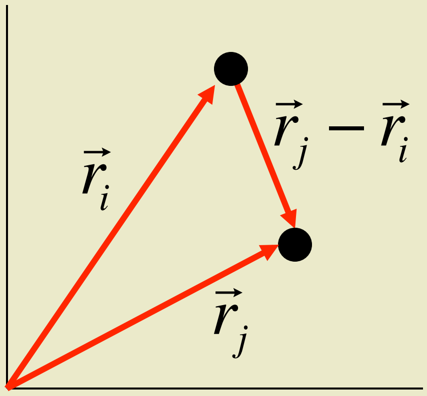
$$= 2\sum_i \frac{1}{2} m \cdot \vec{v}_i^{\,2} \quad \boxed{= \ 2K}$$

(2 x total kinetic energy)

The Virial Theorem

A: $$\sum_i \frac{d\vec{p}_i}{dt}\cdot\vec{r}_i = \sum_i \frac{d}{dt}\left(m\cdot\dot{\vec{r}}_i\right)\cdot\vec{r} = \sum_i m\cdot\ddot{\vec{r}}_i\cdot\vec{r}_i = \sum_i \vec{F}_i\cdot\vec{r}_i$$

$\vec{F}_i$ is the sum of all gravitational forces on particle *i* $= \sum_{j\neq i}\vec{F}_{ij}$



$$\vec{F}_{ij} = G\frac{m_i m_j}{r_{ij}^2}\frac{\left(\vec{r}_j-\vec{r}_i\right)}{r_{ij}} = -G\frac{m_i m_j}{r_{ij}^3}\left(\vec{r}_i-\vec{r}_j\right)$$

$$\sum_i \vec{F}_i\cdot\vec{r}_i = \sum_i\sum_{j\neq i}\vec{F}_{ij}\cdot\vec{r}_i = \cdots+\vec{F}_{ij}\cdot\vec{r}_i+\vec{F}_{ji}\cdot\vec{r}_j+\cdots$$

$$\left(\vec{F}_{ij}=-\vec{F}_{ji}\right) \qquad = \cdots+\vec{F}_{ij}\cdot\left(\vec{r}_i-\vec{r}_j\right)+\cdots = \sum_i\sum_{j>i}\vec{F}_{ij}\cdot\left(\vec{r}_i-\vec{r}_j\right)$$

$$= \sum_i\sum_{j>i}-G\frac{m_i m_j}{r_{ij}^3}\cdot\left(\vec{r}_i-\vec{r}_j\right)^2 = \sum_i\sum_{j>i}-G\frac{m_i m_j}{r_{ij}} \boxed{=\ \Omega}$$

gravitational
potential
energy

The Virial Theorem

$$\boxed{\frac{1}{2}\frac{d^2I}{dt^2} = 2K + \Omega}$$

In a globally static system: $\frac{d^2I}{dt^2} = 0 \longrightarrow \boxed{2K = -\Omega}$

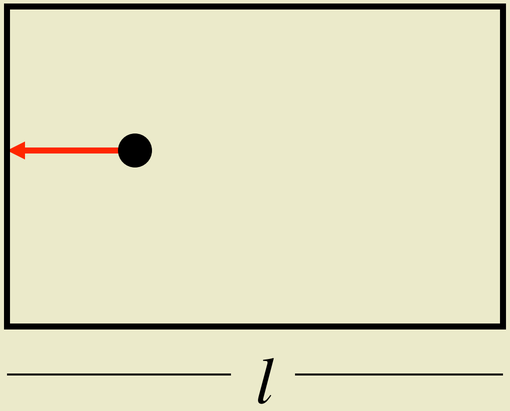
For a collisionless system: $W = K + \Omega \longrightarrow \boxed{W = \frac{1}{2}\Omega = -K}$

bound!

What about a collisional system? How does the 2*K* term relate to the internal energy of a gas?

The Virial Theorem

<u>Kinetic theory</u>: pressure in a gas is caused by the sum of collisions of particles and momentum transfer



Particle *i* colliding with wall in x-direction

$$\Delta p_i = 2p_{x,i} = 2m_i v_{x,i}$$

Time between collisions of same particle with wall

$$\Delta t = \frac{2l}{v_{x,i}}$$

Force due to particle i

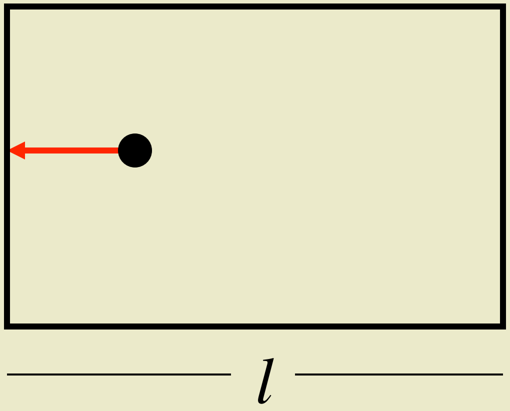
$$F_i = \frac{\Delta p_i}{\Delta t} = \frac{m_i v_{x,i}^2}{l}$$

Total force on wall

$$F = \sum_i \frac{m_i v_{x,i}^2}{l} = \frac{1}{l} \sum_i m_i v_{x,i}^2$$

The Virial Theorem

Kinetic theory: pressure in a gas is caused by the sum of collisions of particles and momentum transfer



Total force on all 6 walls

$$F = \frac{2}{l}\sum_i m_i\left(v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2\right) = \frac{2}{l}\sum_i m_i v_i^2$$

Total force on 1 wall

$$F = \frac{1}{3l}\sum_i m_i v_i^2 \quad = \frac{2}{3l}K$$

Pressure

$$P = \frac{F}{A} = \frac{2K}{3V} \longrightarrow \boxed{2K = 3P \cdot V}$$

The Virial Theorem

In reality, pressure is changing over volume
(can think of our box as a volume element *dV*)

$$2K = 3\int_V P dV$$

In a star's spherical shells

$$dm = \rho dV$$

$$\boxed{2K = \int_M \frac{3P}{\rho} dm}$$

Depends on the equation of state

For a relation btw *P* and *E* of $P = (\gamma - 1)\rho E$ (γ-law equation of state)

$$2K = \int_M 3(\gamma - 1) E dm = 3(\gamma - 1)\int_M E dm \quad = 3(\gamma - 1)U$$

K is only equal to U if $\gamma = \frac{5}{3}$ ⟶ i.e., for an ideal monoatomic gas

The Virial Theorem

Virial theorem: $\boxed{3(\gamma-1)U+\Omega=0}$

Total energy

$$W=\Omega+U$$

$$=\Omega-\frac{\Omega}{3(\gamma-1)}$$

$$=\Omega\left(1-\frac{1}{3(\gamma-1)}\right)$$

$$\boxed{=\Omega\frac{3\gamma-4}{3(\gamma-1)}}$$

Total energy must be < 0 for star to be stable. Otherwise, it has enough energy to disperse itself.

$$\rightarrow\gamma>\frac{4}{3}$$

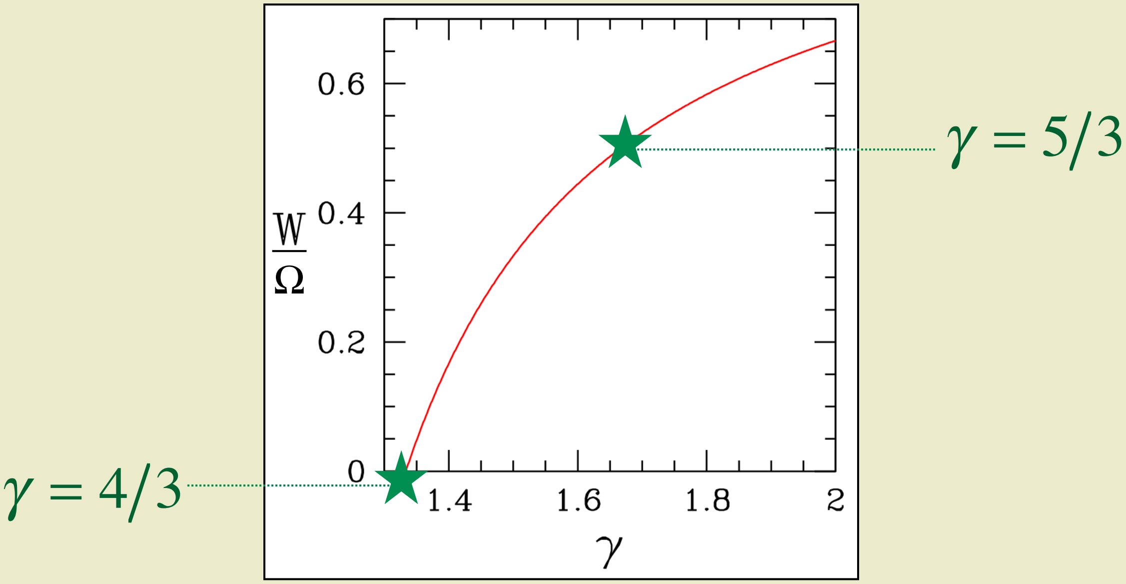
- Ideal gas $\left(\gamma=\frac{5}{3}\right)$: safely bound
- Radiation $\left(\gamma=\frac{4}{3}\right)$: unstable

As radiation pressure becomes more important, star becomes less bound.

The Virial Theorem

Virial theorem: $\boxed{3(\gamma - 1)U + \Omega = 0}$

$$W = \Omega \frac{3\gamma - 4}{3(\gamma - 1)}$$



$\gamma = 5/3$

$\gamma = 4/3$

As a star radiates, it loses energy (*without other energy sources*).

$$\dot{W} < 0 \;\rightarrow\; \begin{cases} \dot{\Omega} < 0 \\ \dot{U} > 0 \end{cases} \;\longrightarrow$$ star contracts and heats up

The Virial Theorem: Applications

Internal temperature

$$\boxed{3(\gamma-1)U+\Omega=0}$$

- For an ideal monoatomic gas: $\gamma = \frac{5}{3} \rightarrow U = -\frac{\Omega}{2}$
- Also, the energy density is: $E = \frac{3}{2}nkT \rightarrow U = \frac{3}{2}nkTV$

$$\left\{ \begin{array}{ll} n: & \text{number density of particles} \\ \mu: & \text{mean molecular weight} \\ N_A: & \text{Avogadro's number} \end{array} \right\} n = \frac{\rho N_A}{\mu}$$

$$U = \frac{3}{2}\frac{\rho N_A}{\mu}kTV = \frac{3}{2}\frac{N_A}{\mu}kMT \rightarrow T = \frac{2}{3}\frac{\mu}{N_A k}\frac{U}{M}$$

- Virial theorem: $U = -\frac{\Omega}{2} = \frac{q}{2}\frac{GM^2}{R}$

The Virial Theorem: Applications

Internal temperature

$$T = \frac{q}{3}\frac{\mu G}{N_A k}\frac{M}{R} \longrightarrow \boxed{T = 5\times 10^6\,K\left(\frac{q}{3/5}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}}$$

(actual central temperature of sun is 15 million K)

- Internal T >> surface T ➔ strong T gradient
- Most of the sun is highly ionized
- T is sufficiently high for nuclear fusion (~1 million K)
- Fusion happens due to gravitational energy

Timescales

Dynamical timescale

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega}$$

- Virial theorem: $\dfrac{d^2 I}{dt^2} \approx \dfrac{GM^2}{R}$

$$\left.\begin{array}{l} \dfrac{I}{t_{\mathrm{dyn}}^2} \approx \dfrac{GM^2}{R} \quad \rightarrow \quad t_{\mathrm{dyn}} \approx \left(\dfrac{R \cdot I}{GM^2}\right)^{1/2} \\ I \approx MR^2 \end{array}\right\} \rightarrow t_{\mathrm{dyn}} \approx \left(\frac{R^3}{GM}\right)^{1/2} \approx (G\rho)^{-1/2}$$

- Escape velocity: $v_{\mathrm{esc}}^2 \sim \dfrac{GM}{R}$
- Timescale for collapse: $t_{\mathrm{dyn}} \sim \dfrac{R}{v_{\mathrm{esc}}} \sim \dfrac{R}{\left(\dfrac{GM}{R}\right)^{1/2}}$

$$\boxed{t_{\mathrm{dyn}} \sim (G\rho)^{-1/2}}$$

For sun, t_{dyn}~ 1 hour

Timescales

Kelvin-Helmholtz timescale

Timescale for star to radiate away its thermal/gravitational energy.

- Star contracts slowly maintaining hydrostatic equilibrium

$$\begin{array}{c} \Delta R < 0 \\ \big\downarrow \\ \Delta\Omega < 0 \end{array} \qquad \left. \begin{array}{c} \Omega = -q\dfrac{GM^2}{R} \\[2ex] \dfrac{d\Omega}{dR} = q\dfrac{GM^2}{R^2} \end{array} \right\} \rightarrow \ \Delta\Omega = +q\frac{GM^2}{R^2}\Delta R$$

- Virial theorem: $\gamma = \frac{5}{3} \ \rightarrow \ W = \frac{\Omega}{2} \ \rightarrow \ \Delta W < 0$

- Energy is lost from the system: radiation

$$\Delta W = \frac{\Delta\Omega}{2}$$

half goes to increasing *U* and half goes to luminosity

Timescales

Kelvin-Helmholtz timescale

Timescale for star to radiate away its thermal/gravitational energy.

- Suppose contraction is solely responsible for maintaining luminosity

$$L = -\frac{dW}{dt} = -\frac{d}{dt}\left(\frac{\Omega}{2}\right) = -\frac{1}{2}\frac{d\Omega}{dR}\frac{dR}{dt} = -\frac{q}{2}\frac{GM^2}{R^2}\frac{dR}{dt}$$

$$\frac{dR}{dt} \approx -\frac{R}{t_{\mathrm{KH}}}$$

$$\boxed{t_{\mathrm{KH}} = \frac{q}{2}\frac{GM^2}{LR}}$$

$$\boxed{t_{\mathrm{KH}} = 2\times10^7\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^2\left(\frac{L}{L_\odot}\right)^{-1}\left(\frac{R}{R_\odot}\right)^{-1}}$$

- Too short a timescale to have happened.

Timescales

Nuclear timescale

Timescale for star to burn its H fuel and leave the Main Sequence.

$$t_{\mathrm{nuc}} = \frac{\text{total available energy}}{\text{luminosity}}$$

$$t_{\mathrm{nuc}} = \frac{\varepsilon f_{\mathrm{core}} M c^2}{L}$$

$\varepsilon =$ efficiency of nuclear burning ($4\,{}^1H \rightarrow {}^4He$) ~ 0.007

$f_{\mathrm{core}} =$ mass fraction of core (star leaves main sequence after core burns) ~ 0.1

For sun, t_{nuc}~ 10 billion years

For stars of solar mass or greater:

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5}$$

$$t_{\mathrm{nuc}} \approx 10^{10}\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^{-2.5}$$

Timescales

$$t_{\text{dyn}} \sim 1 \text{ hour}$$

$$t_{\text{KH}} \sim 20 \text{ million years}$$

$$t_{\text{nuc}} \sim 10 \text{ billion years}$$

$$t_{\text{dyn}} \ll t_{\text{KH}} \ll t_{\text{nuc}}$$

It is valid to assume that stars on the main sequence are in dynamical and thermal equilibrium.

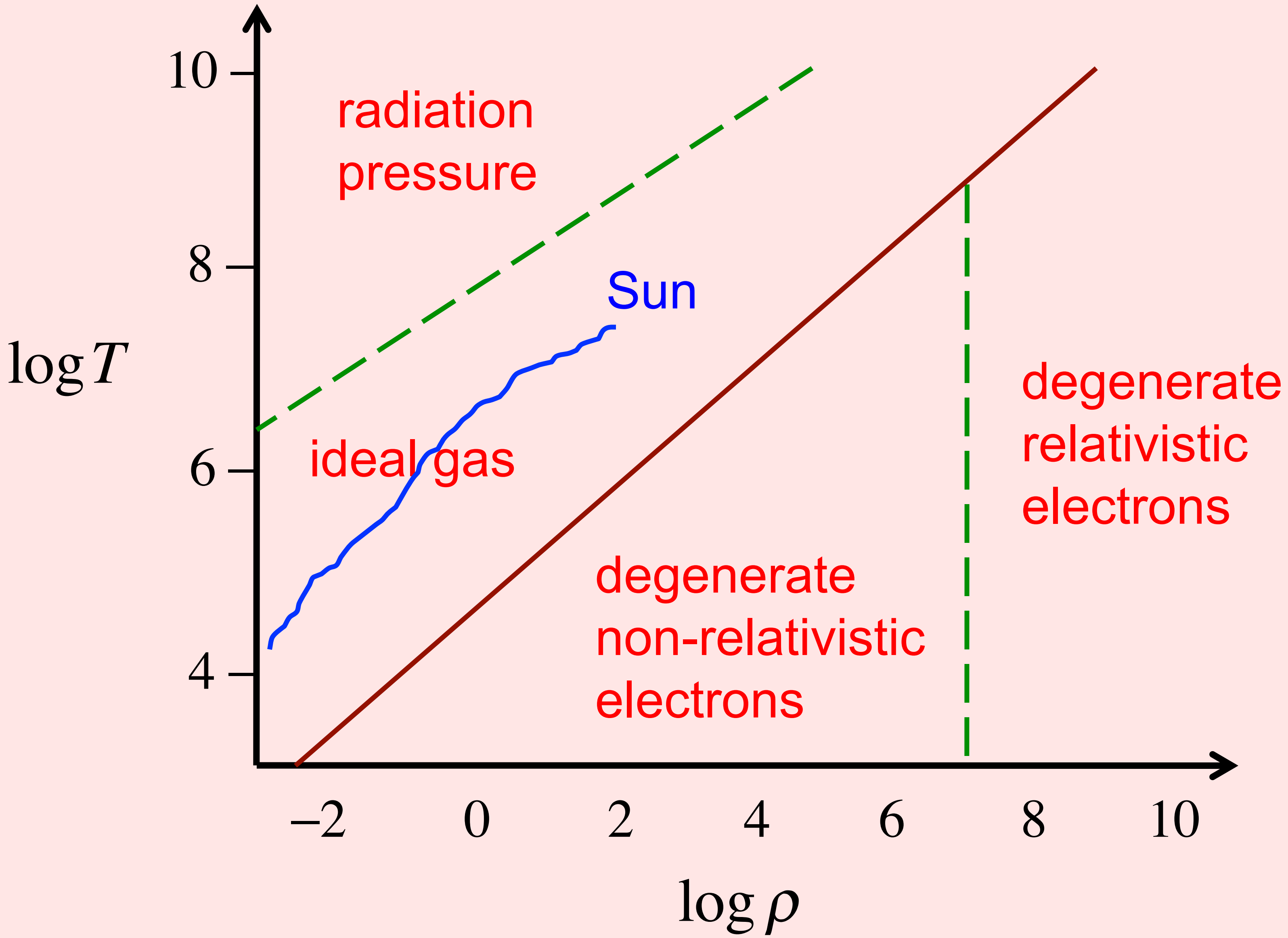
Equation of State

$$P = f\left(\rho, T, X_i\right)$$

To derive the E.O.S., we need to consider:

- Quantum statistics (bosons vs. fermions)
- Non-relativistic vs. relativistic particles
- Degenerate vs. non-degenerate matter

Equation of State



Equation of State

Radiation pressure and degenerate relativistic electrons

$$\gamma = \frac{4}{3} \quad \text{(star is not bound)}$$

This sets limits on
the masses of stars

For most normal stars, ideal gas + radiation pressure are the important contributions

$$P = P_{\text{gas}} + P_{\text{rad}} \qquad = nkT + \frac{1}{3}aT^4$$

n is the number density of all atoms and free electrons

Mean Molecular Weight

To calculate the number density *n*, we need to know:

- constituents of plasma
- state of ionization

Available particles: ions + free electrons

$$n = n_I + n_e = \sum_i \left(n_{I,i} + n_{e,i} \right)$$

Each species *i* :

A_i : nuclear mass number

Z_i : nuclear charge

X_i : fraction by mass

e.g., $A_{^1H} = 1, \; A_{^4He} = 4, \; A_{^{12}C} = 12$

usually $Z_i = \frac{A_i}{2}$ for metals

Mean Molecular Weight

<u>1 mole</u> = amount that has as many atoms of substance as there are atoms of ${}^{12}C$ in $12g$ of ${}^{12}C$.

Avogadro's number: $N_A = 6 \times 10^{23}$

e.g.,

$$\left.\begin{array}{llr} 1 \text{ mole of } {}^{12}C & \text{weighs} & 12g \\ 1 \text{ mole of } {}^{1}H & \text{weighs} & 1g \\ 1 \text{ mole of } {}^{4}He & \text{weighs} & 4g \end{array}\right\} \approx A_i$$

$$\frac{A_i}{N_A} = \frac{\text{mass of mole}}{\#\text{ of atoms per mole}} = \text{mass per atom}$$

Mean Molecular Weight

Ions

$$n_{I,i} = \frac{\left(\text{mass/unit volume}\right) \text{ of } i}{\left(\text{mass of 1 ion}\right) \text{ of } i} \quad = \frac{\rho X_i}{A_i / N_A} = \frac{\rho N_A X_i}{A_i}$$

$$n_I = \sum_i n_{I,i} = \rho N_A \sum_i \frac{X_i}{A_i}$$

$$\frac{\mu_I}{N_A} = \frac{\text{mass of mole for a mixture of ions}}{\text{\# of atoms per mole}} = \text{ mean mass per atom}$$

$$\boxed{n_I = \frac{\rho N_A}{\mu_I}}$$

$$\boxed{\mu_I^{-1} = \sum_i \frac{X_i}{A_i}}$$

Mean Molecular Weight

electrons

also need to know ionization fraction y_i

$$n_{e,i} = n_{I,i} \cdot Z_i \cdot y_i \quad = \rho N_A \frac{X_i}{A_i} Z_i y_i$$

$$y_i = \begin{cases} 0: \text{ completely neutral} \\ 1: \text{ completely ionized} \end{cases}$$

$$n_e = \sum_i n_{e,i} = \rho N_A \sum_i \frac{X_i Z_i y_i}{A_i}$$

$$\boxed{n_e = \frac{\rho N_A}{\mu_e}}$$

$$\boxed{\mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}}$$

Mixture of ions + electrons

μ is the mean weight of particles in units of ^{1}H

$$\boxed{n = \frac{\rho N_A}{\mu}}$$

$$\boxed{\mu^{-1} = \mu_I^{-1} + \mu_e^{-1}}$$

Mean Molecular Weight

$$\left({}^{1}H,{}^{4}He,\text{ metals}\right)\equiv\left(X,Y,Z\right)$$

Composition of Sun

$$\left.\begin{array}{ll} {}^{1}H & \sim 0.71 \\ {}^{4}He & \sim 0.27 \end{array}\right.$$

$$\left.\begin{array}{ll} {}^{16}O & \sim 0.01 \\ {}^{12}C & \sim 0.004 \\ {}^{56}Fe & \sim 0.0015 \\ {}^{28}Si & \sim 0.001 \\ {}^{14}N & \sim 0.001 \\ {}^{24}Mg & \sim 0.0008 \\ {}^{20}Ne & \sim 0.0006 \\ {}^{32}S & \sim 0.0004 \end{array}\right\}$$

These were fused inside stars

In the core

$$\begin{array}{ll} {}^{1}H & \sim 0.34 \\ {}^{4}He & \sim 0.64 \end{array}$$

Mean Molecular Weight

$$\boxed{\mu_I^{-1} = \sum_i \frac{X_i}{A_i}}$$

$$\boxed{\mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}}$$

$$\boxed{\mu^{-1} = \mu_I^{-1} + \mu_e^{-1}}$$

- Fully ionized gas with zero metals (Z=0, y_i=1)

$$\mu_I^{-1} = \frac{X}{A_H} + \frac{Y}{A_{He}} \quad = \frac{X}{1} + \frac{Y}{4} \quad = X + \frac{1-X}{4} \quad \boxed{= \frac{1+3X}{4}}$$

$$\mu_e^{-1} = \frac{X \cdot Z_H \cdot y_H}{A_H} + \frac{Y \cdot Z_{He} \cdot y_{He}}{A_{He}} \quad = \frac{X \cdot 1 \cdot 1}{1} + \frac{Y \cdot 2 \cdot 1}{4} \quad = X + \frac{1-X}{2}$$

$$\boxed{= \frac{1+X}{2}}$$

$$\mu^{-1} = \frac{1+3X}{4} + \frac{1+X}{2} \boxed{= \frac{3+5X}{4}}$$

$$\boxed{X = 0.7 \rightarrow \mu = 0.62}$$

Mean Molecular Weight

$$\boxed{\mu_I^{-1} = \sum_i \frac{X_i}{A_i}} \qquad \boxed{\mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}} \qquad \boxed{\mu^{-1} = \mu_I^{-1} + \mu_e^{-1}}$$

- In the core, all *H* is converted to *He* (X=0)

$$\boxed{X = 0 \rightarrow \mu = 1.3}$$

- Adding metals (fully ionized)

$$\mu_I^{-1} = \frac{X}{A_H} + \frac{Y}{A_{He}} + \frac{Z}{A_Z} \quad = X + \frac{Y}{4} + \frac{Z}{A_Z}$$

$$\mu_e^{-1} = \frac{X \cdot Z_H \cdot y_H}{A_H} + \frac{Y \cdot Z_{He} \cdot y_{He}}{A_{He}} + \frac{Z \cdot \frac{A_Z}{2} \cdot y_Z}{A_Z} \quad = X + \frac{Y}{2} + \frac{Z}{2}$$

$$\mu^{-1} = 2X + \frac{3}{4}Y + \cancelto{0}{\frac{Z}{A_Z}} + \frac{Z}{2} \qquad \boxed{= 2X + \frac{3}{4}Y + \frac{1}{2}Z}$$

$$\boxed{\begin{array}{l} X = 1 \rightarrow \mu = 0.5 \\ Z = 1 \rightarrow \mu = 2 \end{array}}$$

Equation of State

Equations of state may be derived assuming Local Thermodynamic Equilibrium (LTE)

At any position in the star, thermodynamic equilibrium holds locally even though it does not hold globally.

Particle-particle and photon-particle mean free paths are short relative to other length/time scales.

For example, the pressure scale height (height over which the pressure changed by a factor of *e*) is much longer than the mean free path.

$$\lambda_P = -\left(\frac{d\ln P}{dr}\right)^{-1} = -\left(\frac{1}{P}\frac{dP}{dr}\right)^{-1} = \frac{P}{g\rho} \quad \sim R \text{ in most of star}$$

Compare that to the mean free path $\lambda_\gamma \sim 1\mathrm{cm}$

Distribution Functions

Statistical mechanics gives us the phase-space density of a particle species.

$$\boxed{n(p)=\frac{1}{h^3}\sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j+\varepsilon(p)-\mu\right)/kT\right]\pm 1}}$$

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3$$

- $p=|\vec{p}|$: momentum
- g_j : degeneracy of state *j* (# of states having same energy)
- ε_j : energy of state *j* relative to some reference level
- $\varepsilon(p)$: kinetic energy $\varepsilon(p)=\left(p^2c^2+m^2c^4\right)^{1/2}-mc^2$

NR: $\boxed{pc \ll mc^2}$ $$=\left(1+\frac{p^2c^2}{m^2c^4}\right)^{1/2}mc^2-mc^2 \approx \left(1+\frac{1}{2}\frac{p^2c^2}{m^2c^4}\right)mc^2-mc^2 = \frac{p^2}{2m}$$

UR: $\boxed{pc \gg mc^2}$ $$=\left(p^2c^2+\cancel{m^2c^4}\right)^{1/2}-\cancel{mc^2} \approx pc$$

- $\mu=\left(\dfrac{\partial E}{\partial N}\right)$: chemical potential $\displaystyle\sum_i \mu_i dN_i = 0$

$$\text{e.g., } H^+ + e^- \rightarrow H^0 + \gamma$$

$$1\mu_{H^+} + 1\mu_{e^-} - 1\mu_{H^0} = 0$$

- $\pm$: + for fermions (Fermi-Dirac), - for bosons (Bose-Einstein)

Distribution Functions

We can get equation of state quantities from *n(p)*

Integrate over all phase space assuming spherical symmetry

- Space density

$$\boxed{n = \int_0^\infty n(p) 4\pi p^2 \, dp}$$

- Internal energy

  (per unit volume)

$$\boxed{E = \int_0^\infty \varepsilon(p) n(p) 4\pi p^2 \, dp}$$

- Pressure

$$\boxed{P = \frac{1}{3}\int_0^\infty (p \cdot v) n(p) 4\pi p^2 \, dp}$$

Kinetic theory

$$P \cdot V = \frac{1}{3}\sum_i m_i v_i^2 \rightarrow P = \frac{1}{3V}\sum_i p_i v_i$$

Blackbody Radiation

$$\boxed{n(p)=\frac{1}{h^3}\sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j+\varepsilon(p)-\mu\right)/kT\right]\pm 1}}$$

For photons:

- $g=2$ (2 polarization states)
- $\varepsilon_\gamma=0$ (no excited states)
- $\varepsilon(p)=pc$ (fully relativistic)
- $\mu_\gamma=0$
- bosons, so “-”

$$\boxed{n(p)=\frac{1}{h^3}\frac{2}{\exp\left(pc/kT\right)-1}}$$

Blackbody Radiation

Density

$$\boxed{n(p)=\frac{1}{h^3}\frac{2}{\exp(pc/kT)-1}}$$

$$\frac{pc}{kT}=x$$

$$dp=\frac{kT}{c}dx$$

$$p^2=\left(\frac{kT}{c}\right)^2x^2$$

$$n_\gamma=\int_0^\infty n(p)4\pi p^2\,dp \qquad =\frac{8\pi}{h^3}\int_0^\infty\frac{p^2dp}{\exp(pc/kT)-1}$$

$$=\frac{8\pi}{h^3}\left(\frac{kT}{c}\right)^3\boxed{\int_0^\infty\frac{x^2dx}{e^x-1}}\quad\approx 2.4$$

$$=2.4\frac{8\pi}{h^3}\left(\frac{kT}{c}\right)^3 \qquad \boxed{=20.28T^3\text{cm}^{-3}}$$

Blackbody Radiation

Energy

$$\boxed{n(p) = \frac{1}{h^3} \frac{2}{\exp(pc/kT) - 1}}$$

$$E_\gamma = \int_0^\infty (pc)\, n(p)\, 4\pi p^2\, dp \;= \frac{8\pi c}{h^3} \int_0^\infty \frac{p^3 dp}{\exp(pc/kT) - 1}$$

$$\frac{pc}{kT} = x$$

$$dp = \frac{kT}{c} dx$$

$$p^3 = \left(\frac{kT}{c}\right)^3 x^3$$

$$= \frac{8\pi c}{h^3} \left(\frac{kT}{c}\right)^4 \boxed{\int_0^\infty \frac{x^3 dx}{e^x - 1}} \quad = \frac{\pi^4}{15}$$

$$= \frac{\pi^4}{15} \frac{8\pi c}{h^3} \left(\frac{kT}{c}\right)^4 \quad = \left(\frac{8\pi^5 k^4}{15 h^3 c^3}\right) T^4 \qquad \boxed{= aT^4}$$

$$a = 7.5 \times 10^{-15}\,\mathrm{erg\ cm^{-3} K^{-4}}$$

Blackbody Radiation

Pressure

$$\boxed{n(p) = \frac{1}{h^3} \frac{2}{\exp(pc/kT) - 1}}$$

$$P_\gamma = \frac{1}{3} \int_0^\infty (pc) n(p) 4\pi p^2 \, dp \quad = \frac{1}{3} E_\gamma \quad \boxed{= \frac{1}{3} a T^4}$$

$$P = (\gamma - 1) E \quad = \frac{1}{3} E \quad \rightarrow \gamma = \frac{4}{3}$$

Blackbody Radiation

Spectrum

$$\boxed{n(p)=\frac{1}{h^3}\frac{2}{\exp(pc/kT)-1}}$$

The energy density of photons with momentum between *p* and *p*+*dp* is

$$\varepsilon(p)n(p)4\pi p^2dp \qquad =\frac{8\pi}{h^3}\frac{(pc)p^2dp}{\exp(pc/kT)-1}$$

The energy density of photons with frequency between *v* and *v*+*dv* is

$$\left(\begin{array}{c} pc=h\nu \\ dp=(h/c)d\nu \end{array}\right) \qquad \frac{8\pi}{h^3}\frac{h\nu(h\nu/c)^2(h/c)d\nu}{\exp(h\nu/kT)-1}$$

$$\boxed{B_\nu d\nu=\frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT)-1}d\nu}$$

$$\left(B_\nu : \mathrm{erg\ cm^{-3}\ Hz^{-1}}\right)$$

Planck function

Ideal Monoatomic Gas

$$\boxed{n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j + \varepsilon(p) - \mu\right)/kT\right] \pm 1}}$$

For atoms:

- $\varepsilon_j = \varepsilon_0$ (single energy state)
- $\varepsilon(p) = \dfrac{p^2}{2m}$ (non-relativistic)
- $\mu/kT \ll -1$ (+/-1 term can be neglected)

$$n(p) = \frac{1}{h^3} \frac{g}{\exp\left[\left(\varepsilon_0 + p^2/2m - \mu\right)/kT\right]}$$

$$\boxed{n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}}$$

Ideal Monoatomic Gas

Density

$$n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}$$

$$n = \int_0^\infty n(p) 4\pi p^2 \, dp \qquad = \underbrace{\frac{4\pi g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT}}_{A} \int_0^\infty e^{-p^2/2mkT} p^2 \, dp$$

$$= A \int_0^\infty e^{-x} (2mkTx) \frac{mkT}{(2mkTx)^{1/2}} dx$$

$$= \frac{A}{2} (2mkT)^{3/2} \int_0^\infty x^{1/2} e^{-x} \, dx \quad = \frac{\sqrt{\pi}}{2}$$

$$= A \frac{\pi^{1/2}}{4} (2mkT)^{3/2}$$

$$= \frac{g}{h^3} (2\pi mkT)^{3/2} e^{\mu/kT} e^{-\varepsilon_0/kT}$$

$$p^2/2mkT = x$$

$$dp = \frac{mkT}{p} dx$$

$$p = (2mkT)^{1/2} x^{1/2}$$

Ideal Monoatomic Gas

Pressure

$$\boxed{n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}}$$

$$P = \frac{1}{3}\int_0^\infty (pv)\, n(p)\, 4\pi p^2\, dp = \underbrace{\frac{1}{3}\frac{4\pi g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT}}_{A} \int_0^\infty \frac{p^2}{m} e^{-p^2/2mkT} p^2\, dp$$

$$= A\frac{1}{3m}\int_0^\infty e^{-x}(2mkTx)^2 \frac{mkT}{(2mkTx)^{1/2}}dx$$

$$= \frac{1}{3m}\frac{A}{2}(2mkT)^{5/2}\int_0^\infty x^{3/2}e^{-x}\,dx$$

$$= \frac{1}{3m}\frac{A}{2}(2mkT)^{5/2}\left[\cancelto{0}{-x^{3/2}e^{-x}}\Big|_0^\infty + \frac{3}{2}\int_0^\infty x^{1/2}e^{-x}\,dx\right]$$

$$= kT\frac{A}{2}(2mkT)^{3/2}\int_0^\infty x^{1/2}e^{-x}\,dx \quad \boxed{= nkT}$$

$$p^2/2mkT = x$$

$$dp = \frac{mkT}{p}dx$$

$$p = (2mkT)^{1/2}x^{1/2}$$

$$u = x^{3/2} \qquad du = (3/2)x^{1/2}dx$$

$$v = -e^{-x} \qquad dv = e^{-x}$$

Ideal Monoatomic Gas

$$\boxed{n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}}$$

Energy

$$E = \int_0^\infty \left(p^2/2m\right) n(p) 4\pi p^2 \, dp \qquad = \frac{3}{2} P \qquad \boxed{= \frac{3}{2} nkT}$$

$$P = (\gamma - 1) E \qquad = \frac{2}{3} E \qquad \rightarrow \gamma = \frac{5}{3}$$

Degenerate Fermions

$$\boxed{n(p)=\frac{1}{h^3}\sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j+\varepsilon(p)-\mu\right)/kT\right]\pm 1}}$$

For fermions:

- $g=2$ (2 spin states)
- $\varepsilon_0=mc^2$ (no excited states)
- $\varepsilon(p)=\left(p^2c^2+m^2c^4\right)^{1/2}-mc^2$
  $=mc^2\left(\sqrt{1+\left(p/mc\right)^2}-1\right)$
- fermions, so “+”

$$\boxed{n(p)=\frac{2}{h^3}\frac{1}{\exp\left[\left(\varepsilon(p)-\left(\mu-mc^2\right)\right)/kT\right]+1}}$$

Degenerate Fermions

$$\boxed{n(p) = \frac{2}{h^3} \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right) / kT\right] + 1}}$$

A completely degenerate gas behaves as if $T \to 0$

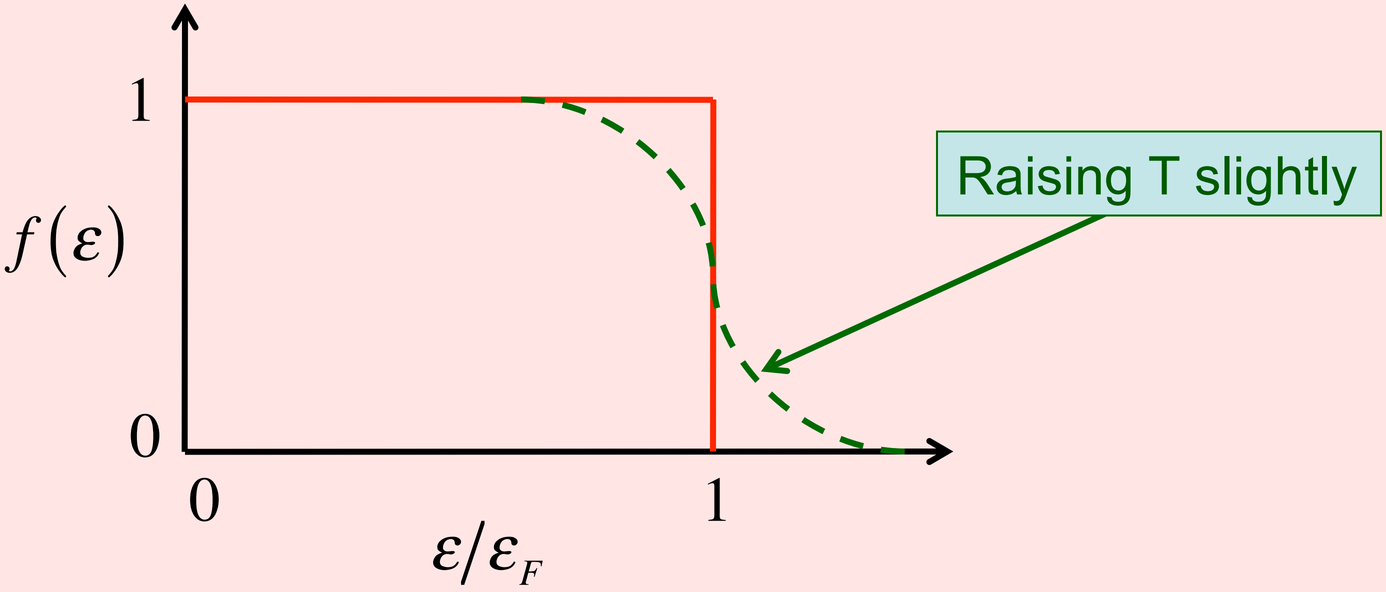
The probability that an energy state is occupied is

$$f(\varepsilon) = \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right) / kT\right] + 1} = \begin{cases} 1 \quad \text{if} \quad \varepsilon < \left(\mu - mc^2\right) \\ 0 \quad \text{if} \quad \varepsilon > \left(\mu - mc^2\right) \end{cases}$$

Critical energy: Fermi energy $\varepsilon_F = \mu - mc^2$

Particles cannot have greater energy than this.

Degenerate Fermions



Raising T slightly

Fermi momentum p_F

$$x = \frac{p}{mc} \rightarrow x_F = \frac{p_F}{mc} \qquad \varepsilon_F = mc^2\left(\sqrt{1+x_F^2}-1\right)$$

Chemical potential is thus $\mu_F = mc^2 + \varepsilon_F$

This is the maximum total energy of particles

Degenerate Fermions

Density

$$n(p) = \frac{2}{h^3} \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right) / kT\right] + 1}$$

$$n = \int_0^\infty n(p) 4\pi p^2 \, dp \qquad = \frac{8\pi}{h^3} \int_0^{p_F} p^2 \, dp \qquad = \frac{8\pi}{h^3} \frac{p_F^3}{3}$$

$$= \frac{8\pi}{3} \left(\frac{mc}{h} \right)^3 x_F^3$$

For electrors: $n_e = \frac{8\pi}{3} \left(\frac{m_e c}{h} \right)^3 x_F^3 = 5.9 \times 10^{29} x_F^3 \ \mathrm{cm}^{-3}$

Degenerate Fermions

Pressure

$$n(p)=\frac{2}{h^3}\frac{1}{\exp\left[\left(\varepsilon(p)-\left(\mu-mc^2\right)\right)/kT\right]+1}$$

$$P=\frac{1}{3}\int_0^\infty (pv)n(p)4\pi p^2\,dp$$

$$=\frac{8\pi}{3h^3}\int_0^{p_F} v\cdot p^3\,dp$$

$$=\frac{8\pi}{3mh^3}\int_0^{p_F}\frac{p^4dp}{\sqrt{1+(p/mc)^2}}$$

$$\varepsilon(p)=mc^2\left(\sqrt{1+(p/mc)^2}-1\right)$$

$$v=\frac{d\varepsilon}{dp}$$

$$=mc^2\frac{1}{2}\left(1+\left(\frac{p}{mc}\right)^2\right)^{-1/2}\cdot 2\frac{p}{mc}\cdot\frac{1}{mc}$$

$$=\frac{p}{m}\left[1+\left(\frac{p}{mc}\right)^2\right]^{-1/2}$$

Degenerate Fermions

Pressure

$$P = \frac{8\pi}{3mh^3}\int_0^{p_F}\frac{p^4\,dp}{\sqrt{1+(p/mc)^2}}$$

$$\frac{p}{mc} = x$$

$$dp = mc\,dx$$

$$p^4 = (mc)^4 x^4$$

$$= \frac{8\pi}{3mh^3}\int_0^{x_F}\frac{(mc)^5 x^4\,dx}{\sqrt{1+x^2}} \quad = \frac{8\pi m^4 c^5}{3h^3}\int_0^{x_F}\frac{x^4\,dx}{\sqrt{1+x^2}}$$

$$\boxed{= \frac{\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2\left[x_F\left(2x_F^2-3\right)\left(1+x_F^2\right)^{1/2}+3\sinh^{-1}x_F\right]}$$

Degenerate Fermions

Energy

$$n(p) = \frac{2}{h^3}\frac{1}{\exp\left[\left(\varepsilon(p)-\left(\mu - mc^2\right)\right)/kT\right]+1}$$

$$E = \int_0^\infty \varepsilon(p) n(p) 4\pi p^2\,dp \quad = \frac{8\pi}{h^3}\int_0^{p_F} mc^2\left(\sqrt{1+(p/mc)^2}-1\right)p^2\,dp$$

$$= \frac{8\pi}{h^3}mc^2\int_0^{x_F}\left(\sqrt{1+x^2}-1\right)(mc)^3 x^2\,dx$$

$$= 8\pi\left(\frac{mc}{h}\right)^3 mc^2\int_0^{x_F}\left(\sqrt{1+x^2}-1\right)x^2\,dx$$

$$\frac{p}{mc} = x$$

$$dp = mcdx$$

$$p^2 = (mc)^2 x^2$$

$$= \frac{\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2\left[8x_F^3\left(\sqrt{1+x_F^2}-1\right)-x_F\left(2x_F^2-3\right)\left(1+x_F^2\right)^{1/2}-3\sinh^{-1}x_F\right]$$

Degenerate Fermions

$$n = \underbrace{\frac{8\pi}{3}\left(\frac{mc}{h}\right)^3}_{A} x_F^3 \quad \boxed{= A x_F^3}$$

$$P = \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2 \int_0^{x_F} \frac{x^4 dx}{\sqrt{1+x^2}} \quad \boxed{= Amc^2 \int_0^{x_F} \frac{x^4 dx}{\sqrt{1+x^2}}}$$

$$E = 8\pi\left(\frac{mc}{h}\right)^3 mc^2 \int_0^{x_F} \left(\sqrt{1+x^2}-1\right) x^2\, dx \quad \boxed{= 3Amc^2 \int_0^{x_F} \left(\sqrt{1+x^2}-1\right) x^2\, dx}$$

Degenerate Fermions

$$n = A x_F^3$$

$$P = Amc^2 \int_0^{x_F} \frac{x^4 dx}{\sqrt{1+x^2}}$$

$$E = 3Amc^2 \int_0^{x_F} \left(\sqrt{1+x^2} - 1\right) x^2\, dx$$

Look at limiting cases: NR, UR

NR: $x_F \ll 1$

$$P \approx Amc^2 \int_0^{x_F} x^4\, dx \quad = Amc^2 \frac{x_F^5}{5} \quad = Amc^2 \frac{1}{5}\left(\frac{n}{A}\right)^{5/3}$$

$$\boxed{= \frac{mc^2}{5} A^{-2/3} n^{5/3}}$$

$$E \approx 3Amc^2 \int_0^{x_F} \left(1 + \frac{1}{2}x^2 - 1\right) x^2\, dx \quad = \frac{3}{2} Amc^2 \int_0^{x_F} x^4\, dx \quad \boxed{= \frac{3}{2} P}$$

$$P = (\gamma - 1) E = \frac{2}{3} E \qquad \rightarrow \gamma = \frac{5}{3}$$

Degenerate Fermions

$$\boxed{n = A x_F^3}$$

$$\boxed{P = Amc^2 \int_0^{x_F} \frac{x^4\,dx}{\sqrt{1+x^2}}}$$

$$\boxed{E = 3Amc^2 \int_0^{x_F} \left(\sqrt{1+x^2}-1\right) x^2\,dx}$$

Look at limiting cases: NR, UR

UR: $x_F \gg 1$

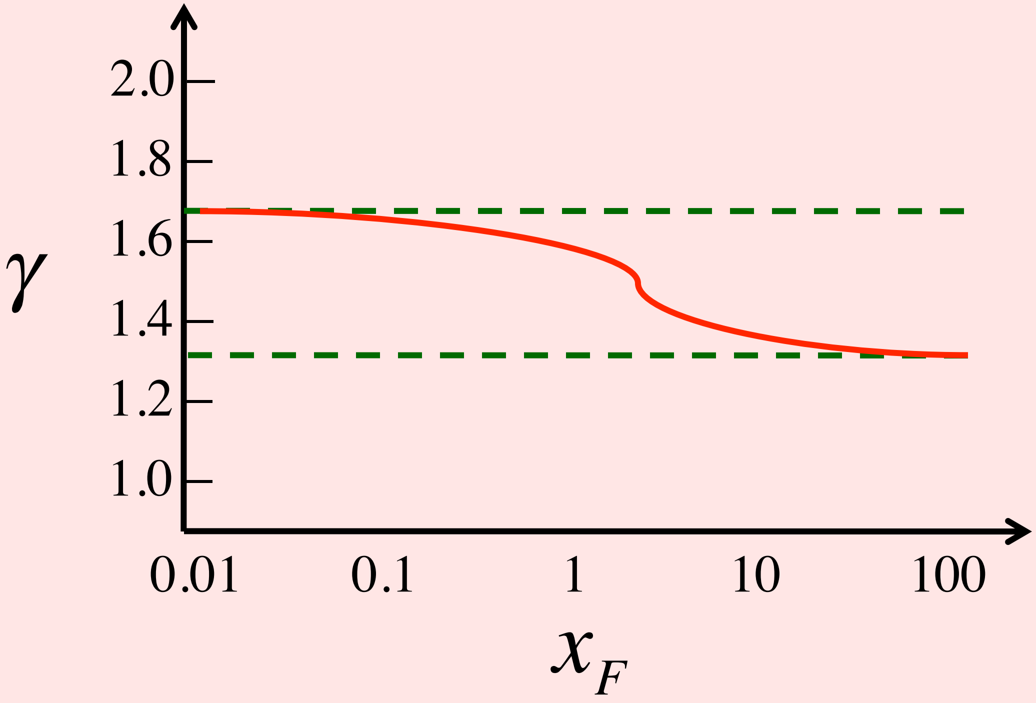
$$P \approx Amc^2 \int_0^{x_F} x^3\,dx = Amc^2 \frac{x_F^4}{4} = Amc^2 \frac{1}{4}\left(\frac{n}{A}\right)^{4/3}$$

$$\boxed{= \frac{mc^2}{4} A^{-1/3} n^{4/3}}$$

$$E \approx 3Amc^2 \int_0^{x_F} (x-1)\,x^2\,dx = 3Amc^2 \int_0^{x_F} x^3\,dx \quad \boxed{= 3P}$$

$$P = (\gamma - 1)E = \frac{1}{3}E \quad \rightarrow \gamma = \frac{4}{3}$$

Degenerate Fermions



- Completely degenerate gas has same gamma as ideal gas.
- In the relativistic limit, it behaves like radiation.

Degenerate Fermions

$$\boxed{n = \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 x_F^3}$$

Convert to mass density: $n = \dfrac{N_A \rho}{\mu}$

e.g., electrons:

$$n_e = \frac{N_A \rho}{\mu_e} = \frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3 x_F^3$$

$$\frac{\rho}{\mu_e} = \frac{8\pi}{3N_A}\left(\frac{m_e c}{h}\right)^3 x_F^3 \quad \boxed{= 9.7 \times 10^5 x_F^3 \text{ g cm}^{-3}}$$

e.g., neutrons:

$$n_n = \frac{N_A \rho}{\mu_n} = \frac{8\pi}{3}\left(\frac{m_n c}{h}\right)^3 x_F^3$$

$$\frac{\rho}{\mu_n} = \frac{8\pi}{3N_A}\left(\frac{m_n c}{h}\right)^3 x_F^3 \quad \boxed{= 6.1 \times 10^{15} x_F^3 \text{ g cm}^{-3}}$$

Degenerate Fermions

Useful approximations

- Partly degenerate gas

$$P_e \approx \left(P_{e,nd}^2 + P_{e,d}^2\right)^{1/2}$$

- Partly relativistic gas

$$P_{e,d} \approx \left(P_{e,d,NR}^{-2} + P_{e,d,UR}^{-2}\right)^{-1/2}$$

This formula picks out the smallest pressure

These interpolation formulae are good to ~2%

Equation of State Summary

- Start with $\rho, \ T, \ X_i$ at some point in star
- Assume Local Thermodynamic Equilibrium
- Total gas pressure is the sum of components

$$\boxed{P = P_{\text{rad}} + P_{\text{ion}} + P_e}$$

- The radiation pressure is

$$\boxed{P_{\text{rad}} = \frac{1}{3} a T^4} \quad \left(a = 7.5 \times 10^{-15} \, \text{erg cm}^{-3} \text{K}^{-4} \right)$$

- The ion ideal gas pressure is

$$\boxed{P_{\text{ion}} = \frac{N_A k}{\mu_I} \rho T} \quad \left(N_A = 6.022 \times 10^{23} \, \text{mole}^{-1} \right)$$

Equation of State Summary

- The electron pressure can be a mixture of non-degenerate and degenerate pressure

$$\boxed{P_e \approx \left(P_{e,nd}^2 + P_{e,d}^2\right)^{1/2}}$$

- Where the electron non-degenerate pressure is

$$\boxed{P_{e,nd} = \frac{N_A k}{\mu_e} \rho T}$$

- And the electron degenerate pressure can be non-relativistic or relativistic

$$\boxed{P_{e,d} \approx \left(P_{e,d,NR}^{-2} + P_{e,d,UR}^{-2}\right)^{-1/2}}$$

Equation of State Summary

- Where the non-relativistic, degenerate pressure is

$$\boxed{P_{e,d,NR} = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e}\left(\frac{N_A}{\mu_e}\right)^{5/3} \rho^{5/3}}$$

- And the ultra-relativistic, degenerate pressure is

$$\boxed{P_{e,d,UR} = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4}\left(\frac{N_A}{\mu_e}\right)^{4/3} \rho^{4/3}}$$

- Compute the mean molecular weights from X_i

$$\boxed{\mu_I^{-1} = \sum_i \frac{X_i}{A_i}} \qquad \boxed{\mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}}$$

- Get the ionization fractions from the Saha equation.

The Boltzmann Equation

Ideal gas : $\boxed{n = \frac{g}{h^3}\left(2\pi mkT\right)^{3/2} e^{\mu/kT} e^{-\varepsilon_0/kT}}$

Consider two species of particles: atoms in different energy levels that can be excited or de-excited via photons.

e.g., $\boxed{H_1 + \gamma \rightarrow H_2}$ $1\mu_{H_1} + 0 = 1\mu_{H_2}$

$$\frac{n_1}{n_2} = \frac{\left(g_1/h^3\right)\cancel{\left(2\pi mkT\right)^{3/2}}\ \cancel{e^{\mu_1/kT}}\, e^{-\varepsilon_1/kT}}{\left(g_2/h^3\right)\cancel{\left(2\pi mkT\right)^{3/2}}\ \cancel{e^{\mu_2/kT}}\, e^{-\varepsilon_2/kT}} = \frac{g_1 e^{-\varepsilon_1/kT}}{g_2 e^{-\varepsilon_2/kT}}$$

$$\boxed{\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{-(\varepsilon_1-\varepsilon_2)/kT}}$$

The Boltzmann Equation

Total number of atoms: $n_{\text{tot}} = \sum_i n_i$

$$\frac{n_{\text{tot}}}{n_i} = \frac{n_1}{n_i} + \frac{n_2}{n_i} + \cdots \quad = \frac{g_1}{g_i} e^{-\varepsilon_1/kT} e^{\varepsilon_i/kT} + \frac{g_2}{g_i} e^{-\varepsilon_2/kT} e^{\varepsilon_i/kT} + \cdots$$

$$= \frac{1}{g_i} e^{\varepsilon_i/kT} \left(g_1 e^{-\varepsilon_1/kT} + g_2 e^{-\varepsilon_2/kT} + \cdots \right) \quad = \frac{1}{g_i e^{-\varepsilon_i/kT}} \sum_j g_j e^{-\varepsilon_j/kT}$$

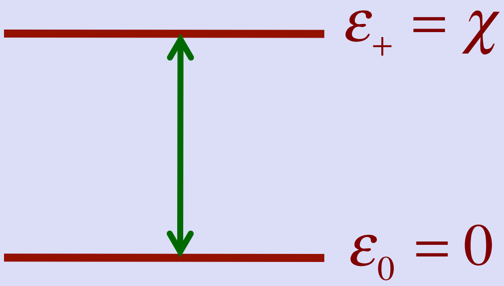
$$\boxed{\frac{n_i}{n_{\text{tot}}} = \frac{g_i}{G} e^{-\varepsilon_i/kT}}$$

Partition function: $\boxed{G = \sum_j g_j e^{-\varepsilon_j/kT}}$

Like an effective statistical weight for all states

The Saha Equation

Consider an atomic gas that is partly ionized



e.g., $\boxed{H^+ + e^- \leftrightarrow H^0 + \gamma}$ $\quad 1\mu^+ + 1\mu^- = 1\mu^0$

- charged ions in ground state $n^+ = \dfrac{g^+}{h^3}\left(2\pi m_p kT\right)^{3/2} e^{\mu^+/kT} e^{-\chi/kT}$
- free electrons $n^- = \dfrac{g^-}{h^3}\left(2\pi m_e kT\right)^{3/2} e^{\mu^-/kT}$
- neutral ions in ground state $n^0 = \dfrac{g^0}{h^3}\left(2\pi m_p kT\right)^{3/2} e^{\mu^0/kT}$

Combine to eliminate chemical potentials

$$\boxed{\frac{n^+ n^-}{n^0} = \frac{g^+ g^-}{g^0} \frac{\left(2\pi m_e kT\right)^{3/2}}{h^3} e^{-\chi/kT}}$$

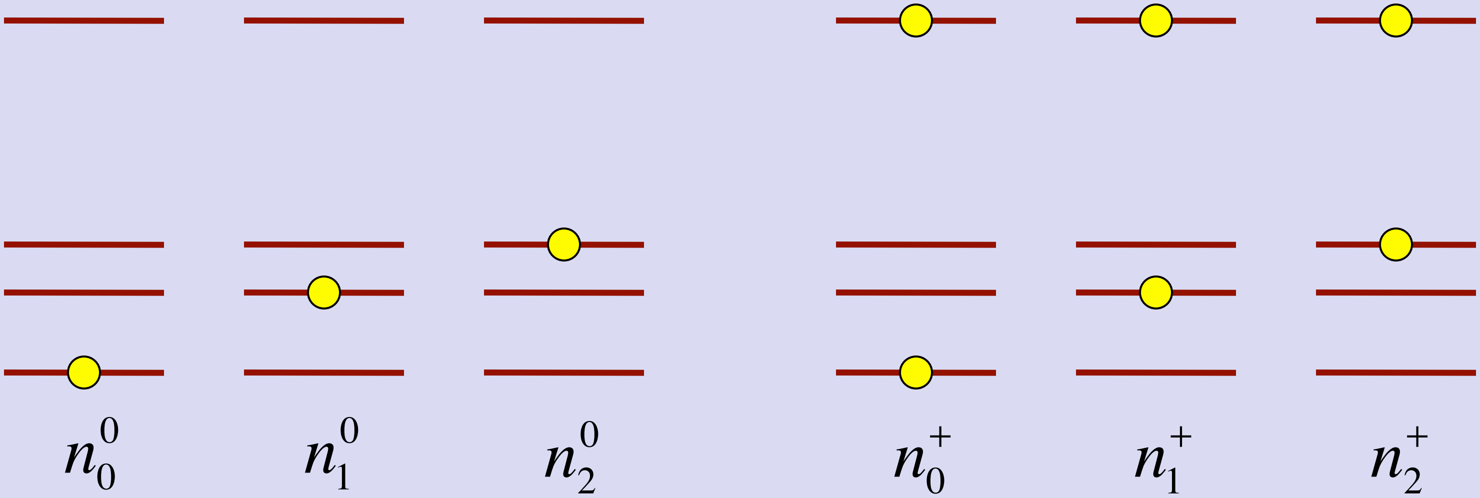
The Saha Equation

$$\frac{n^+ n^-}{n^0} = \frac{g^+ g^-}{g^0} \frac{\left(2\pi m_e kT\right)^{3/2}}{h^3} e^{-\chi/kT}$$

This is only for atoms in the ground state.

In general, we have neutral and ionized atoms in multiple states

Use Boltzmann to generalize to all states of a given atom



$$n^0 = \sum_i n_i^0 \qquad n^+ = \sum_i n_i^+$$

The Saha Equation

$$\frac{n^+n^-}{n^0} = \frac{g^+g^-}{g^0}\frac{\left(2\pi m_e kT\right)^{3/2}}{h^3}e^{-\chi/kT}$$

Boltzmann: $$\frac{n_i}{n_{\text{tot}}} = \frac{g_i}{G}e^{-\varepsilon_i/kT}$$

$$\frac{n_0^+n^-}{n_0^0} \qquad \frac{g_0^+g^-}{g_0^0}$$

$$\frac{n_0^0}{n^0} = \frac{g_0^0}{G^0} \qquad \frac{n_0^+}{n^+} = \frac{g_0^+}{G^+}$$

LHS:

$$\frac{n_0^+n^-}{n_0^0} = \frac{n^+\frac{g_0^+}{G^+}n^-}{n^0\frac{g_0^0}{G^0}} = \frac{n^+n^-}{n^0}\frac{G^0}{G^+}\frac{g_0^+}{g_0^0}$$

RHS:

$$= \frac{g_0^+g^-}{g_0^0}\frac{\left(2\pi m_e kT\right)^{3/2}}{h^3}e^{-\chi/kT}$$

$$g^- = 2$$

$$n^- = n_e$$

$$\frac{n^+n_e}{n^0} = \frac{2G^+}{G^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}e^{-\chi/kT}$$

Saha!

Can generalize to: $\frac{n^{j+1}n_e}{n^j}$

The Saha Equation

$$\boxed{\frac{n^+ n_e}{n^0} = \frac{2G^+}{G^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}}$$

Example: pure H

$$y = \frac{n^+}{n_I}$$

$$n_I = n^+ + n^0$$

$$n_e = n^+$$

$$G^0 \approx 2$$

$$G^+ = 1$$

$$\left.\begin{aligned} n^+ &= y n_I \\ n^0 &= (1-y) n_I \\ n_e &= y n_I \end{aligned}\right\} \frac{n^+ n_e}{n^0} = \frac{y n_I \cdot y n_I}{(1-y) n_I} = \frac{y^2}{1-y} n_I$$

$$\boxed{\frac{y^2}{1-y} = \frac{1}{n_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}} \qquad n_I = N_A \rho$$

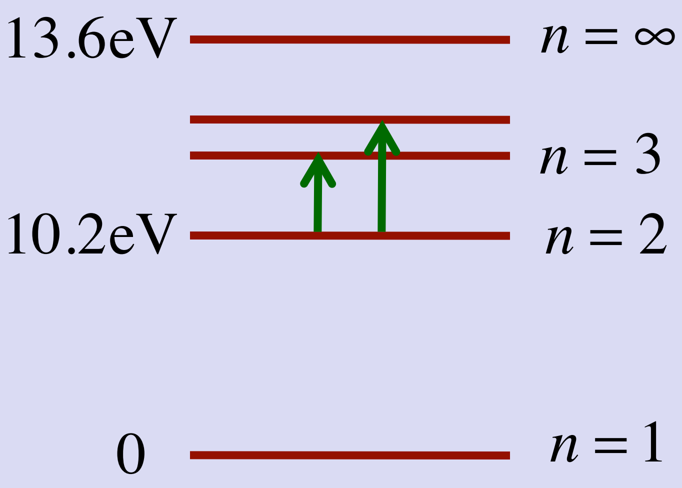
$$\frac{y^2}{1-y} = \frac{4\times 10^{-9}\,\mathrm{g\ cm^{-3}}}{\rho} T^{3/2} e^{-1.578\times 10^5/T}$$

50% ionized at T=10^4K
for low densities

The Saha Equation

Example: Balmer line strength

Strength of Balmer lines depend on fraction of H gas that is in the n=2 excited state.



We want $\dfrac{n_2^0}{n_I}$

Boltzmann gives us $\dfrac{n_2^0}{n^0} = \dfrac{g_2^0}{G^0} e^{-\varepsilon_2/kT}$

$$\left.\begin{aligned} \frac{n_2^0}{n_I} &= \frac{n_2^0}{n^0}\frac{n^0}{n_I} \\ \frac{n^0}{n_I} &= 1-y \end{aligned}\right\}$$

$$\boxed{\frac{n_2^0}{n_I} = \left(1-y\right)\frac{g_2^0}{G^0} e^{-\varepsilon_2/kT}}$$

$$g_2^0 = 8$$

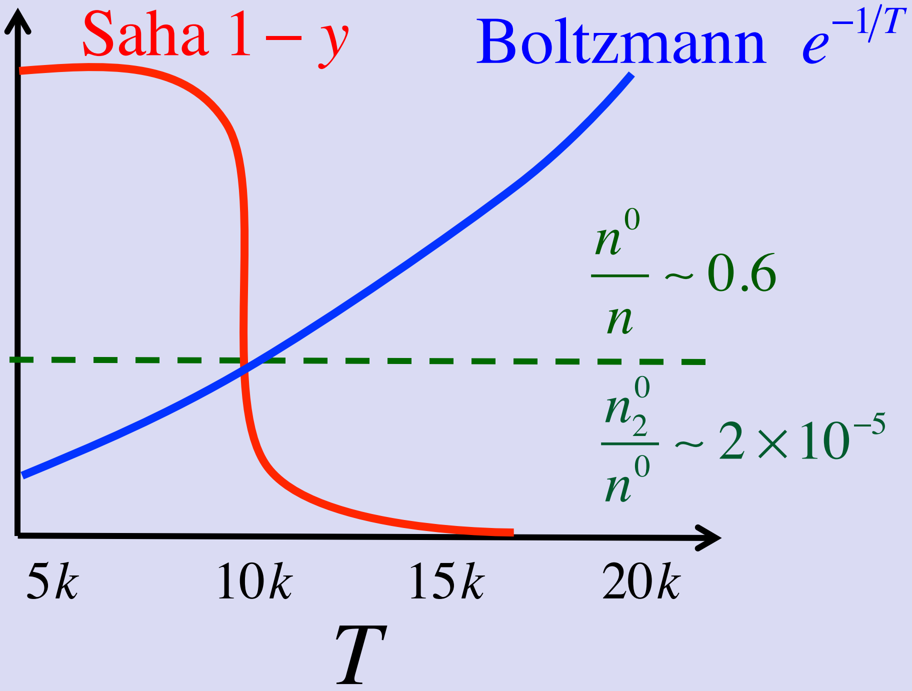
$$G^0 = \sum_i g_i e^{-\varepsilon_i/kT} \approx 2$$

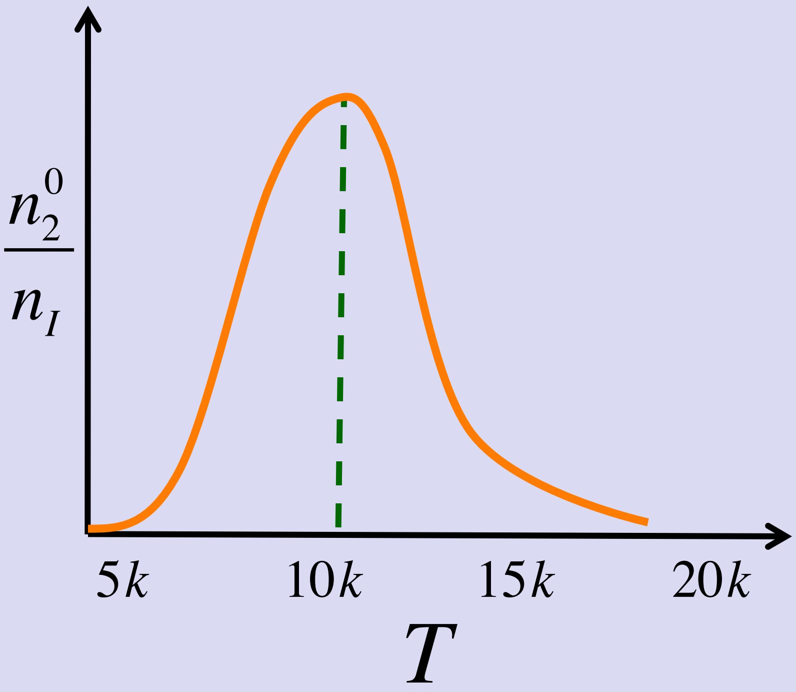
$$\varepsilon_2 = 10.2 \text{ eV}$$

The Saha Equation

$$\frac{n_2^0}{n^0} = (1-y)\frac{g_2^0}{G^0} e^{-\varepsilon_2/kT}$$

$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$





The Saha Equation

A different way to write the Saha equation

$$\boxed{\frac{n^+ n_e}{n^0} = \frac{2G^+}{G^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}}$$

$$\log\left(\frac{n^+ n_e}{n^0}\right) = \log\left(\frac{2G^+}{G^0}\right) + \frac{3}{2}\log\left(\frac{2\pi m_e kT}{h^2}\right) + \log\left(e^{-\chi/kT}\right)$$

$$\log\left(\frac{n^+}{n^0}\right) + \log n_e = \log\left(\frac{2G^+}{G^0}\right) + \frac{3}{2}\log\left(\frac{2\pi m_e k}{h^2}\right) + \frac{3}{2}\log T - \ln 10\,\frac{\chi}{kT}$$

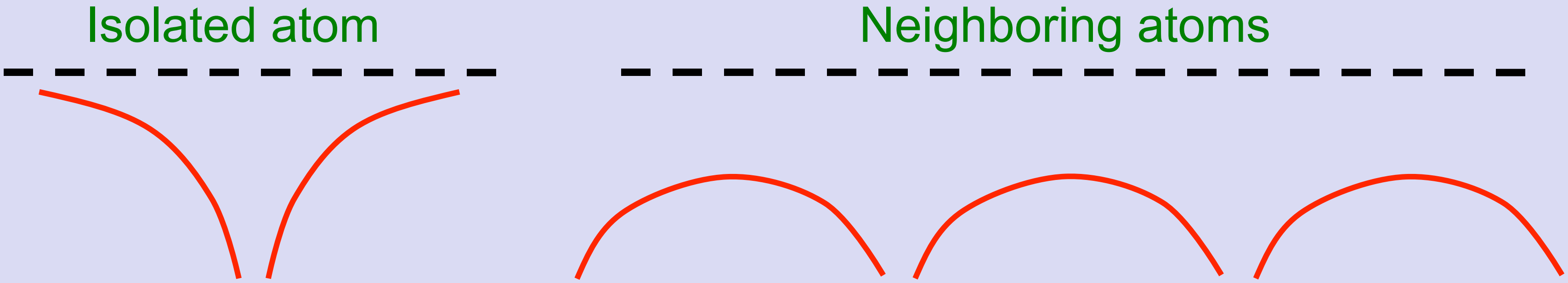
$$P_e = n_e kT \quad \rightarrow \log n_e = \log P_e - \log k - \log T$$

$$\boxed{\log\left(\frac{n^+}{n^0}\right) = \log\left(\frac{2G^+}{G^0}\right) + \frac{5}{2}\log T - \log P_e - \frac{5040\,\chi}{T} - 0.48}$$

The Saha Equation

The Saha equation works best at moderate densities

- At very low densities (e.g., corona), we must worry about whether LTE applies.
- At very high densities, atoms are not isolated.



Overlap of potentials of neighboring atoms lowers the effective ionization energy, which leads to greater ionization than Saha predicts.

For density n and separation a: $n = \left(\frac{4}{3} \pi a^3 \right)^{-1}$

set $a =$ radius of first Bohr orbit of $H = 0.5 \times 10^{-8}$ cm $\boxed{\rightarrow \rho \sim 3 \text{ g cm}^{-3}}$

Above this density, even the ground state is affected and all H is ionized.

The Saha Equation

- For ions of charge Z, the ratio of electrostatic energy to thermal energy is a measure of whether Coulomb effects affect the ideal-ness of an ideal gas.

$$\Gamma_C = \frac{Z^2 e^2 / a}{kT} \qquad \Gamma_C = 1 \rightarrow \rho = 85 \text{ g cm}^{-3} \left(\frac{T}{10^6 K} \right)^3$$

Important in very low mass stars

Equation of State

For pure H

- ideal gas vs. radiation

  $\rho = 1.5 \times 10^{-23} T^3$

- degenerate vs. not

  $\rho = 6 \times 10^{-9} T^{3/2}$

- neutral vs. ionized

  $\rho = 8 \times 10^{-9} T^{3/2} e^{-1.58\times 10^5 / T}$

- pressure ionized

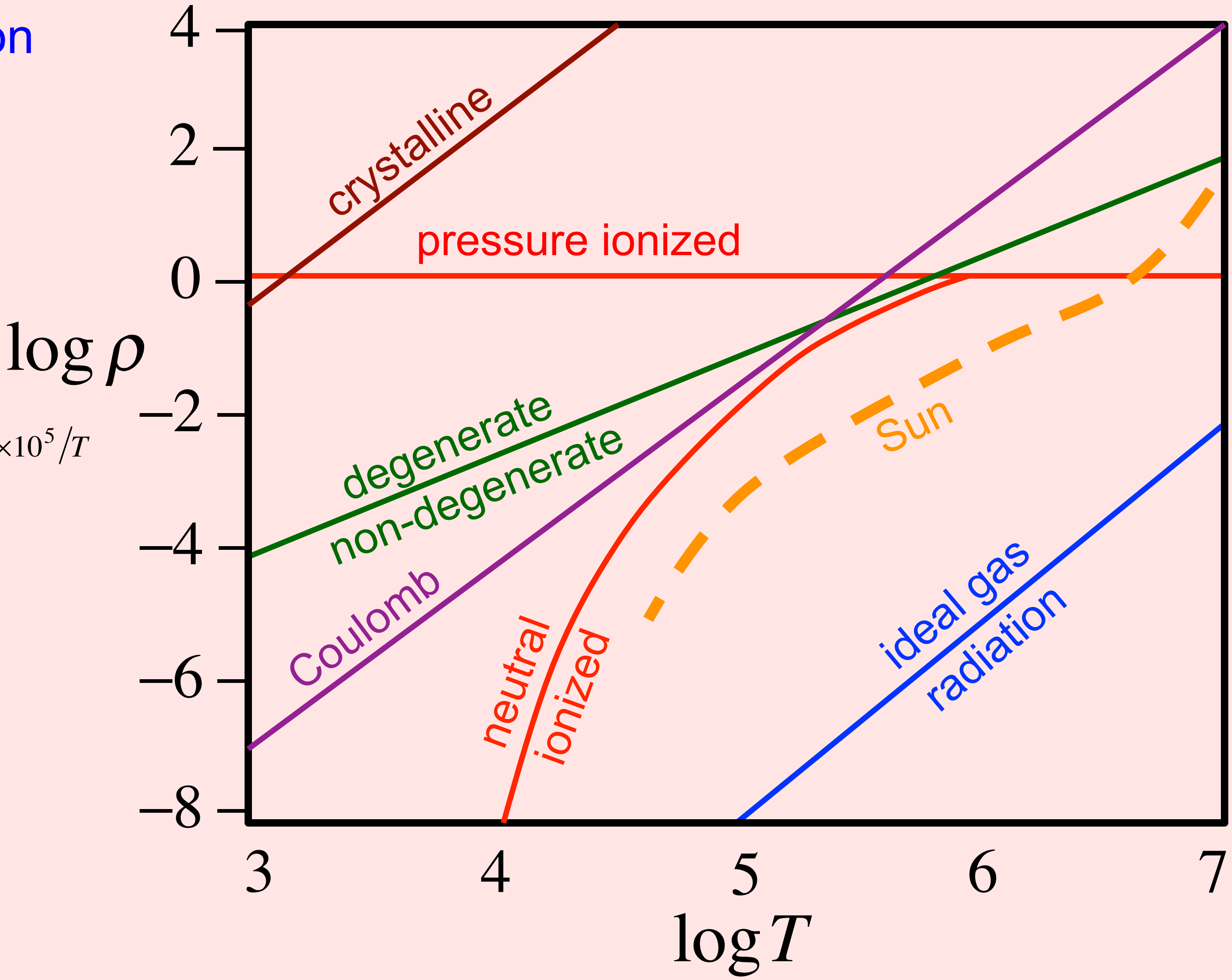
  $\rho \approx 1$

- Coulomb effects

  $\rho = 8.5 \times 10^{-17} T^3$

- crystalline structure

  $\rho = 4.2 \times 10^{-10} T^3$



Adiabatic Exponents

$Q:$ heat $\left(\mathrm{erg\ g^{-1}}\right)$

$V_\rho:$ specific volume $\left(\mathrm{cm^3 g^{-1}}\right) = \frac{1}{\rho}$

$E:$ specific internal energy $\left(\mathrm{erg\ g^{-1}}\right)$

First law of thermodynamics

$$dQ = dE + PdV_\rho \quad = dE + Pd\left(\frac{1}{\rho}\right) \quad = dE - \frac{P}{\rho^2}d\rho$$

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{\mathrm{ad}} \quad = -\left(\frac{\partial \ln P}{\partial \ln V_\rho}\right)_{\mathrm{ad}}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\mathrm{ad}} \quad = \frac{1}{\nabla_{\mathrm{ad}}}$$

$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{\mathrm{ad}} \quad = -\left(\frac{\partial \ln T}{\partial \ln V_\rho}\right)_{\mathrm{ad}}$$

$$\nabla_{\mathrm{ad}} = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{\mathrm{ad}} \left(\frac{\partial \ln \rho}{\partial \ln P}\right)_{\mathrm{ad}}$$

$$= \frac{\Gamma_3 - 1}{\Gamma_1}$$

$$= \frac{\Gamma_2 - 1}{\Gamma_2}$$

Adiabatic Exponents

- Γ_1 describes how pressure responds to compression
  (relevant for dynamical processes like pulsations)
- Γ_2 describes how temperature responds to changes in pressure
  (relevant for determining whether convection takes place)
- Γ_3 describes how temperature responds to compression

Ideal Gas: $\Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{5}{3}$

Radiation: $\Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{4}{3}$

Γ_3 is equivalent to the γ in the γ-law equation of state $P = (\gamma - 1)\rho E$

For mixtures of gas and radiation, as well as for cases where chemical reaction happen, the adiabatic exponents can differ from each other.

Adiabatic Exponents: Ideal gas + Radiation

Pressure

$$P = P_{\text{gas}} + P_{\text{rad}} = \frac{N_A k}{\mu} \rho T + \frac{1}{3} a T^4$$

$$\boxed{= \frac{N_A k}{\mu} \frac{T}{V_\rho} + \frac{1}{3} a T^4}$$

Specific Internal Energy

$$E = E_{\text{gas}} + E_{\text{rad}} = \frac{3}{2} \frac{N_A k}{\mu} T + a \frac{T^4}{\rho}$$

$$\boxed{= \frac{3}{2} \frac{N_A k}{\mu} T + a T^4 V_\rho}$$

Adiabatic Exponents: Ideal gas + Radiation

$$P = \frac{N_A k}{\mu}\frac{T}{V_\rho} + \frac{1}{3}aT^4$$

$$E = \frac{3}{2}\frac{N_A k}{\mu}T + aT^4 V_\rho$$

Adiabatic change
$dQ = 0$

$$dQ = \left(\frac{\partial E}{\partial T}\right)_{V_\rho} dT + \left(\frac{\partial E}{\partial V_\rho}\right)_T dV_\rho + PdV_\rho = 0$$

$$= \left(\frac{3N_A k}{2\mu} + 4aT^3 V_\rho\right)dT + \left(aT^4\right)dV_\rho + \left(\frac{N_A k}{\mu}\frac{T}{V_\rho} + \frac{1}{3}aT^4\right)dV_\rho = 0$$

$$= \left(\frac{3N_A k}{2\mu} + 4aT^3 V_\rho\right)dT + \left(\frac{N_A k}{\mu}\frac{T}{V_\rho} + \frac{4}{3}aT^4\right)dV_\rho = 0$$

$$= \left(\frac{3N_A k}{2\mu}\frac{T}{V_\rho} + 4aT^4\right)\frac{V_\rho}{T}dT + \left(\frac{N_A k}{\mu}\frac{T}{V_\rho} + \frac{4}{3}aT^4\right)dV_\rho = 0$$

$$\boxed{= \left(\frac{3}{2}P_{\text{gas}} + 12P_{\text{rad}}\right)\frac{dT}{T} + \left(P_{\text{gas}} + 4P_{\text{rad}}\right)\frac{dV_\rho}{V_\rho} = 0}$$

Adiabatic Exponents: Ideal gas + Radiation

$$\boxed{P = \frac{N_A k}{\mu}\frac{T}{V_\rho} + \frac{1}{3}aT^4}$$

$$\boxed{E = \frac{3}{2}\frac{N_A k}{\mu}T + aT^4 V_\rho}$$

Equation of state

$$dP = \left(\frac{\partial P}{\partial T}\right)_{V_\rho} dT + \left(\frac{\partial P}{\partial V_\rho}\right)_T dV_\rho = \left(\frac{N_A k}{\mu}\frac{1}{V_\rho} + \frac{4}{3}aT^3\right)dT + \left(-\frac{N_A k}{\mu}\frac{T}{V_\rho^2}\right)dV_\rho$$

$$= \left(\frac{N_A k}{\mu}\frac{T}{V_\rho} + \frac{4}{3}aT^4\right)\frac{1}{T}dT + \left(-\frac{N_A k}{\mu}\frac{T}{V_\rho}\right)\frac{1}{V_\rho}dV_\rho$$

$$= \left(P_{\text{gas}} + 4P_{\text{rad}}\right)\frac{dT}{T} - P_{\text{gas}}\frac{dV_\rho}{V_\rho} \quad = dP = \left(\frac{\partial P}{\partial T}\right)_{\text{ad}} dT \quad = \frac{P}{T}\left(\cancelto{\frac{\Gamma_2}{\Gamma_2 - 1}}{\frac{\partial \ln P}{\partial \ln T}}\right)_{\text{ad}} dT$$

$$\rightarrow \left(P_{\text{gas}} + 4P_{\text{rad}}\right)\frac{dT}{T} - P_{\text{gas}}\frac{dV_\rho}{V_\rho} - \frac{\Gamma_2}{\Gamma_2 - 1}\left(P_{\text{gas}} + P_{\text{rad}}\right)\frac{dT}{T} = 0$$

$$\boxed{= \left(P_{\text{gas}} + 4P_{\text{rad}} - \frac{\Gamma_2}{\Gamma_2 - 1}\left(P_{\text{gas}} + P_{\text{rad}}\right)\right)\frac{dT}{T} - P_{\text{gas}}\frac{dV_\rho}{V_\rho} = 0}$$

Adiabatic Exponents: Ideal gas + Radiation

1st Law of Thermodynamics

$$\left(\frac{3}{2}P_{\text{gas}}+12P_{\text{rad}}\right)\frac{dT}{T}+\left(P_{\text{gas}}+4P_{\text{rad}}\right)\frac{dV_\rho}{V_\rho}=0$$

Equation of State

$$\left(P_{\text{gas}}+4P_{\text{rad}}-\frac{\Gamma_2}{\Gamma_2-1}\left(P_{\text{gas}}+P_{\text{rad}}\right)\right)\frac{dT}{T}-P_{\text{gas}}\frac{dV_\rho}{V_\rho}=0$$

$$\left.\begin{aligned}Ax+By&=0\\Cx+Dy&=0\end{aligned}\right\}\rightarrow\frac{A}{C}=\frac{B}{D}$$

$$\frac{P_{\text{gas}}+4P_{\text{rad}}-\frac{\Gamma_2}{\Gamma_2-1}\left(P_{\text{gas}}+P_{\text{rad}}\right)}{\frac{3}{2}P_{\text{gas}}+12P_{\text{rad}}}=-\frac{P_{\text{gas}}}{P_{\text{gas}}+4P_{\text{rad}}}$$

$$P_{\text{gas}}=\beta P$$
$$P_{\text{rad}}=(1-\beta)P$$

$$\frac{\beta P+4(1-\beta)P-\frac{\Gamma_2}{\Gamma_2-1}P}{\frac{3}{2}\beta P+12(1-\beta)P}=-\frac{\beta P}{\beta P+4(1-\beta)P}$$

Adiabatic Exponents: Ideal gas + Radiation

$$\frac{\beta+4(1-\beta)-\dfrac{\Gamma_2}{\Gamma_2-1}}{\dfrac{3}{2}\beta+12(1-\beta)}=-\frac{\beta}{\beta+4(1-\beta)} \quad\rightarrow\frac{4-3\beta-\dfrac{\Gamma_2}{\Gamma_2-1}}{12-\dfrac{21}{2}\beta}=-\frac{\beta}{4-3\beta}$$

$$\rightarrow\left(4-3\beta-\frac{\Gamma_2}{\Gamma_2-1}\right)(4-3\beta)=(-\beta)\left(12-\frac{21}{2}\beta\right)$$

$$\rightarrow 16-12\beta-12\beta+9\beta^2-\frac{\Gamma_2}{\Gamma_2-1}(4-3\beta)=-12\beta+\frac{21}{2}\beta^2$$

$$\rightarrow\frac{\Gamma_2}{\Gamma_2-1}(4-3\beta)=16-12\beta-\frac{3}{2}\beta^2 \quad\rightarrow\frac{\Gamma_2-1}{\Gamma_2}=\frac{4-3\beta}{16-12\beta-\dfrac{3}{2}\beta^2}$$

$$\rightarrow\frac{1}{\Gamma_2}=1-\frac{4-3\beta}{16-12\beta-\dfrac{3}{2}\beta^2}\quad=\frac{16-12\beta-\dfrac{3}{2}\beta^2-4+3\beta}{16-12\beta-\dfrac{3}{2}\beta^2}\quad=\frac{12-9\beta-\dfrac{3}{2}\beta^2}{16-12\beta-\dfrac{3}{2}\beta^2}$$

Adiabatic Exponents: Ideal gas + Radiation

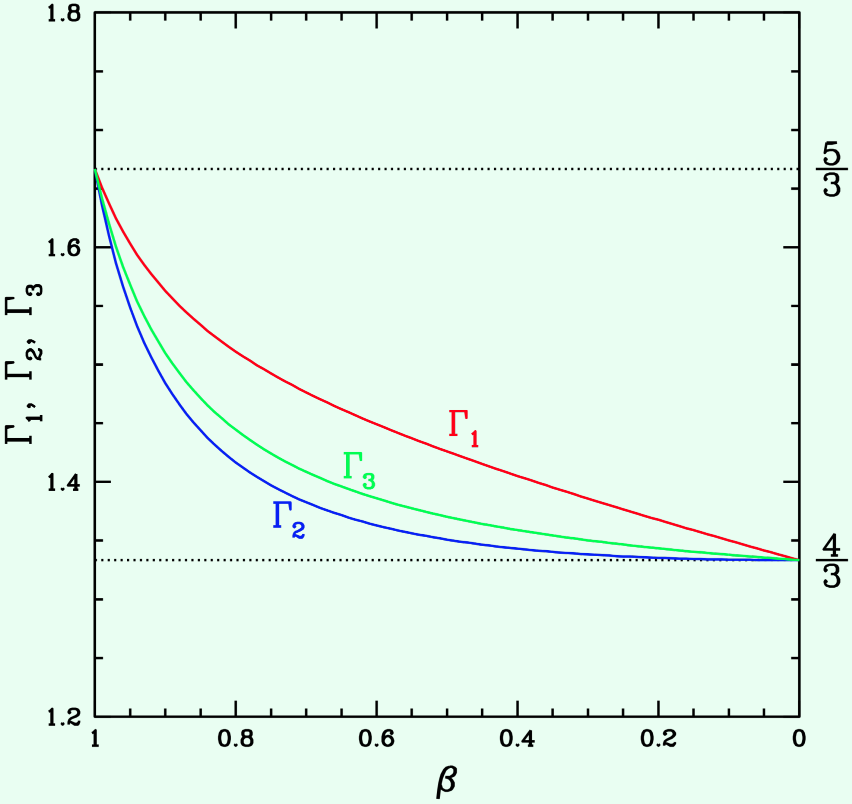
$$\frac{1}{\Gamma_2} = \frac{12 - 9\beta - \frac{3}{2}\beta^2}{16 - 12\beta - \frac{3}{2}\beta^2} \rightarrow \Gamma_2 = \frac{16 - 12\beta - \frac{3}{2}\beta^2}{12 - 9\beta - \frac{3}{2}\beta^2} \rightarrow \Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}$$

Similarly, we get:

$$\Gamma_1 = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta}$$

$$\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}$$

$$\Gamma_3 = \frac{32 - 27\beta}{24 - 21\beta}$$



Adiabatic Exponents: Ionization Zones

Thermodynamics of partially ionized gas is different because

- The number of free particles is not constant
- Ionization energy is required to increase n

Assuming a pure H ideal gas

$$\left.\begin{aligned} n^+ &= y n_I \\ n^0 &= (1-y) n_I \\ n_e &= y n_I \\ n_I &= N_A \rho \end{aligned}\right\} \quad \frac{y^2}{1-y} = \frac{1}{n_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$

$$= \frac{A}{N_A \rho} T^{3/2} e^{-\chi/kT}$$

$$\boxed{\frac{y^2}{1-y} = \frac{A}{N_A} V_\rho T^{3/2} e^{-\chi/kT}}$$

Adiabatic Exponents: Ionization Zones

Pressure

$$P = \left(n^0 + n^+ + n_e\right)kT \quad = \left[\left(1-y\right)n_I + yn_I + yn_I\right]kT$$

$$= \left(1+y\right)n_I kT \quad = \left(1+y\right)N_A \rho kT \rightarrow \quad \boxed{P = \left(1+y\right)N_A k\frac{T}{V_\rho}}$$

Specific Internal Energy

$$E = \frac{3}{2}\left(n^0 + n^+ + n_e\right)\frac{kT}{\rho} + \frac{n^+\chi}{\rho} \qquad = \left(1+y\right)n_I\frac{3}{2}\frac{kT}{\rho} + \frac{yn_I}{\rho}\chi$$

$$\boxed{E = \left(1+y\right)\frac{3}{2}N_A kT + yN_A\chi}$$

Adiabatic Exponents: Ionization Zones

$$\boxed{P = (1+y) N_A k \frac{T}{V_\rho}}$$

$$\boxed{E = (1+y)\frac{3}{2} N_A kT + y N_A \chi}$$

Adiabatic change
$dQ = 0$

$$dQ = \left(\frac{\partial E}{\partial T}\right)_{V_\rho, y} dT + \left(\frac{\partial E}{\partial V_\rho}\right)_{T, y} dV_\rho + \left(\frac{\partial E}{\partial y}\right)_{T, V_\rho} dy + P dV_\rho = 0$$

$$= \left[(1+y)\frac{3}{2} N_A k\right] dT + \left[\frac{3}{2} N_A kT + N_A \chi\right] dy + \left[(1+y) N_A k \frac{T}{V_\rho}\right] dV_\rho = 0$$

$$= \left[(1+y)\frac{3}{2} N_A kT\right]\frac{dT}{T} + \left[(1+y)\frac{3}{2} N_A kT + (1+y) N_A \chi\right]\frac{dy}{1+y}$$

$$+ \left[(1+y) N_A kT\right]\frac{dV_\rho}{V_\rho} = 0$$

$$\boxed{= E_{\text{ideal}}\left[\frac{dT}{T} + \frac{2}{3}\frac{dV_\rho}{V_\rho} + \left(1 + \frac{2}{3}\frac{\chi}{kT}\right)\frac{dy}{1+y}\right] = 0}$$

Adiabatic Exponents: Ionization Zones

$$\boxed{P = (1+y)N_A k \frac{T}{V_\rho}} \qquad \boxed{E = (1+y)\frac{3}{2}N_A kT + yN_A \chi}$$

Equation of State

$$dP = \left(\frac{\partial P}{\partial T}\right)_{V_\rho, y} dT + \left(\frac{\partial P}{\partial V_\rho}\right)_{T,y} dV_\rho + \left(\frac{\partial P}{\partial y}\right)_{T,V_\rho} dy$$

$$= \left[(1+y)N_A k \frac{1}{V_\rho}\right] dT - \left[(1+y)N_A k \frac{T}{V_\rho^2}\right] dV_\rho + \left[N_A k \frac{T}{V_\rho}\right] dy$$

$$= \left[(1+y)N_A k \frac{T}{V_\rho}\right] \frac{dT}{T} - \left[(1+y)N_A k \frac{T}{V_\rho}\right] \frac{dV_\rho}{V_\rho} + \left[(1+y)N_A k \frac{T}{V_\rho}\right] \frac{dy}{1+y}$$

$$dP = (1+y)N_A k \frac{T}{V_\rho}\left[\frac{dT}{T} - \frac{dV_\rho}{V_\rho} + \frac{dy}{1+y}\right] \rightarrow \boxed{dP = P\left[\frac{dT}{T} - \frac{dV_\rho}{V_\rho} + \frac{dy}{1+y}\right]}$$

Adiabatic Exponents: Ionization Zones

$$\boxed{f(y)=\frac{y^2}{1-y}=\frac{A}{N_A}V_\rho T^{3/2}e^{-\chi/kT}}$$

Saha Equation

$$dy=\frac{dy}{df}df=\frac{dy}{df}\left[\left(\frac{\partial f}{\partial T}\right)_{V_\rho}dT+\left(\frac{\partial f}{\partial V_\rho}\right)_T dV_\rho\right]$$

$$=\frac{(1-y)^2}{2y(1-y)+y^2}\left[\frac{A}{N_A}V_\rho\left(\frac{3}{2}T^{1/2}e^{-\chi/kT}+T^{3/2}e^{-\chi/kT}\frac{\chi}{kT^2}\right)dT+\frac{A}{N_A}T^{3/2}e^{-\chi/kT}dV_\rho\right]$$

$$=\frac{(1-y)^2}{2y(1-y)+y^2}\frac{A}{N_A}V_\rho T^{3/2}e^{-\chi/kT}\left[\left(\frac{3}{2}\frac{1}{T}+\frac{\chi}{kT^2}\right)dT+\frac{dV_\rho}{V_\rho}\right]$$

$$=\frac{(1-y)^2}{2y(1-y)+y^2}\frac{y^2}{1-y}\left[\left(\frac{3}{2}+\frac{\chi}{kT}\right)\frac{dT}{T}+\frac{dV_\rho}{V_\rho}\right]$$

Adiabatic Exponents: Ionization Zones

$$dy = \frac{(1-y)^2}{2y(1-y)+y^2}\frac{y^2}{1-y}\left[\left(\frac{3}{2}+\frac{\chi}{kT}\right)\frac{dT}{T}+\frac{dV_\rho}{V_\rho}\right]$$

$$dy = \frac{(1-y)y}{2(1-y)+y}\left[\left(\frac{3}{2}+\frac{\chi}{kT}\right)\frac{dT}{T}+\frac{dV_\rho}{V_\rho}\right]$$

$$dy = \frac{(1-y)y}{2-y}\left[\left(\frac{3}{2}+\frac{\chi}{kT}\right)\frac{dT}{T}+\frac{dV_\rho}{V_\rho}\right]$$

$$\boxed{\frac{dy}{1+y} = \underbrace{\frac{(1-y)y}{(1+y)(2-y)}}_{D(y)}\left[\left(\frac{3}{2}+\frac{\chi}{kT}\right)\frac{dT}{T}+\frac{dV_\rho}{V_\rho}\right]}$$

Adiabatic Exponents: Ionization Zones

1st Law of Thermodynamics

$$\boxed{E_{\text{ideal}}\left[\frac{dT}{T}+\frac{2}{3}\frac{dV_\rho}{V_\rho}+\left(1+\frac{2}{3}\frac{\chi}{kT}\right)\frac{dy}{1+y}\right]=0}$$

Equation of State

$$\boxed{dP=P\left[\frac{dT}{T}-\frac{dV_\rho}{V_\rho}+\frac{dy}{1+y}\right]}$$

Saha

$$\boxed{\frac{dy}{1+y}=D(y)\left[\left(\frac{3}{2}+\frac{\chi}{kT}\right)\frac{dT}{T}+\frac{dV_\rho}{V_\rho}\right]}$$

Adiabatic Exponents: Ionization Zones

1st Law of Thermodynamics

$$E_{\text{ideal}}\left[\frac{dT}{T}+\frac{2}{3}\frac{dV_\rho}{V_\rho}+\left(1+\frac{2}{3}\frac{\chi}{kT}\right)\frac{dy}{1+y}\right]=0$$

$$E_{\text{ideal}}\left[\frac{dT}{T}+\frac{2}{3}\frac{dV_\rho}{V_\rho}+\left(1+\frac{2}{3}\frac{\chi}{kT}\right)D(y)\left[\left(\frac{3}{2}+\frac{\chi}{kT}\right)\frac{dT}{T}+\frac{dV_\rho}{V_\rho}\right]\right]=0$$

$$E_{\text{ideal}}\left[\left[1+\left(1+\frac{2}{3}\frac{\chi}{kT}\right)D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)\right]\frac{dT}{T}+\left[\frac{2}{3}+\left(1+\frac{2}{3}\frac{\chi}{kT}\right)D(y)\right]\frac{dV_\rho}{V_\rho}\right]=0$$

$$\boxed{E_{\text{ideal}}\left[\left[1+D(y)\frac{2}{3}\left(\frac{3}{2}+\frac{\chi}{kT}\right)^2\right]\frac{dT}{T}+\frac{2}{3}\left[1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)\right]\frac{dV_\rho}{V_\rho}\right]=0}$$

Adiabatic Exponents: Ionization Zones

Equation of State

$$\boxed{dP = P\left[\frac{dT}{T} - \frac{dV_\rho}{V_\rho} + \frac{dy}{1+y}\right]}$$

$$dP = P\left[\frac{dT}{T} - \frac{dV_\rho}{V_\rho} + D(y)\left[\left(\frac{3}{2} + \frac{\chi}{kT}\right)\frac{dT}{T} + \frac{dV_\rho}{V_\rho}\right]\right]$$

$$dP = P\left[\left[1 + D(y)\left(\frac{3}{2} + \frac{\chi}{kT}\right)\right]\frac{dT}{T} + \left[D(y) - 1\right]\frac{dV_\rho}{V_\rho}\right]$$

$$dP = \left(\frac{\partial P}{\partial T}\right)_{\text{ad}} dT = \frac{P}{T}\left(\frac{\partial \ln P}{\partial \ln T}\right)_{\text{ad}} dT = \frac{P}{T}\frac{\Gamma_2}{\Gamma_2 - 1} dT$$

$$\frac{\Gamma_2}{\Gamma_2 - 1}\frac{dT}{T} = \left[1 + D(y)\left(\frac{3}{2} + \frac{\chi}{kT}\right)\right]\frac{dT}{T} + \left[D(y) - 1\right]\frac{dV_\rho}{V_\rho}$$

$$\left[1 - \frac{\Gamma_2}{\Gamma_2 - 1} + D(y)\left(\frac{3}{2} + \frac{\chi}{kT}\right)\right]\frac{dT}{T} + \left[D(y) - 1\right]\frac{dV_\rho}{V_\rho} = 0$$

Adiabatic Exponents: Ionization Zones

1st Law

$$\left[1+D(y)\frac{2}{3}\left(\frac{3}{2}+\frac{\chi}{kT}\right)^2\right]\frac{dT}{T}+\frac{2}{3}\left[1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)\right]\frac{dV_\rho}{V_\rho}=0$$

Equation of State

$$\left[1-\frac{\Gamma_2}{\Gamma_2-1}+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)\right]\frac{dT}{T}+\left[D(y)-1\right]\frac{dV_\rho}{V_\rho}=0$$

$$\frac{1-\frac{\Gamma_2}{\Gamma_2-1}+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)}{1+D(y)\frac{2}{3}\left(\frac{3}{2}+\frac{\chi}{kT}\right)^2}=\frac{D(y)-1}{\frac{2}{3}\left[1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)\right]}$$

$$\frac{\Gamma_2}{\Gamma_2-1}=1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)-\frac{\left(D(y)-1\right)\left[1+D(y)\frac{2}{3}\left(\frac{3}{2}+\frac{\chi}{kT}\right)^2\right]}{\frac{2}{3}\left[1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)\right]}$$

Adiabatic Exponents: Ionization Zones

$$\frac{\Gamma_2}{\Gamma_2-1}=1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)-\frac{\left(D(y)-1\right)\left[1+D(y)\frac{2}{3}\left(\frac{3}{2}+\frac{\chi}{kT}\right)^2\right]}{\frac{2}{3}\left[1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)\right]}$$

…algebra…

$$\boxed{\frac{\Gamma_2}{\Gamma_2-1}=\frac{\frac{5}{2}+D(y)\left[\frac{15}{4}+5\frac{\chi}{kT}+\left(\frac{\chi}{kT}\right)^2\right]}{1+D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)}}$$

$$\boxed{D(y)=\frac{y(1-y)}{(1+y)(2-y)}}$$

Similarly:

$$\boxed{\Gamma_3-1=\frac{2+2D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)}{3+2D(y)\left(\frac{3}{2}+\frac{\chi}{kT}\right)^2}}$$

Adiabatic Exponents: Ionization Zones

$$\boxed{\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{1}{\nabla_{\rm ad}} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\rm ad}}$$

Gas heats up

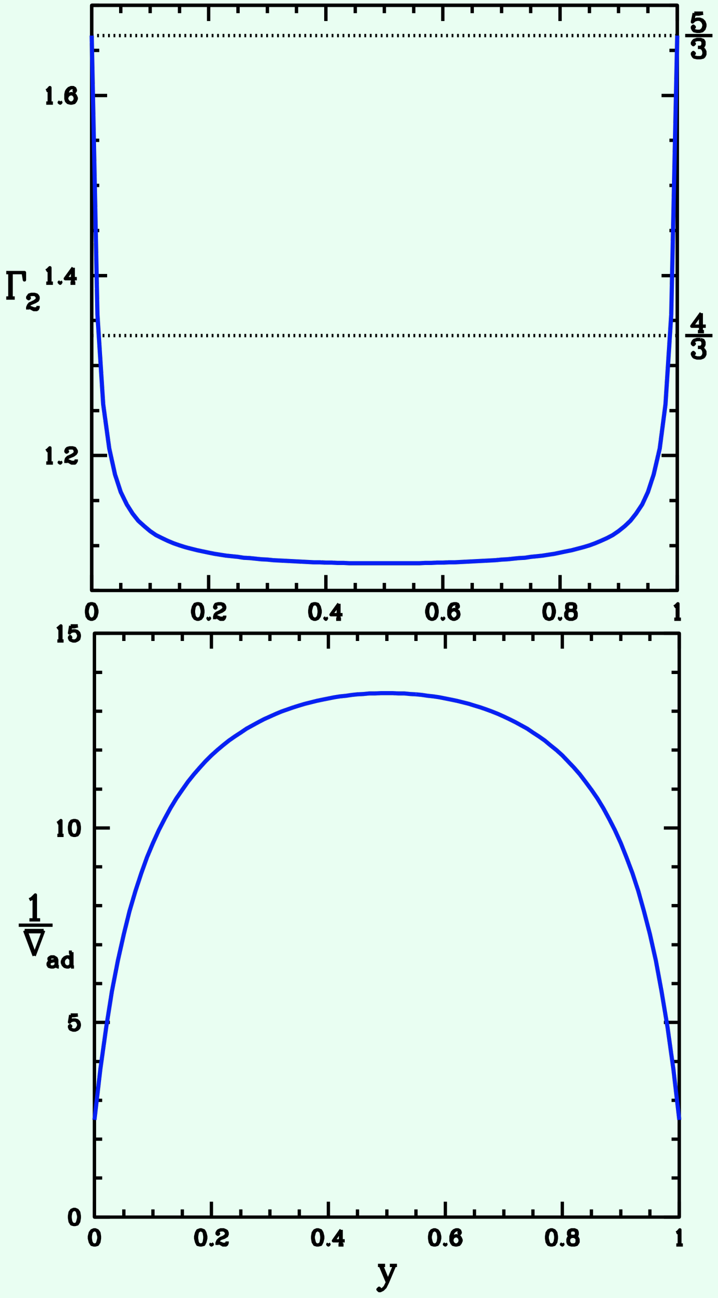
→ rapid increase of ionization

→ increase in number of free particles

→ pressure goes up faster than ~T

Pressure increases faster than normal

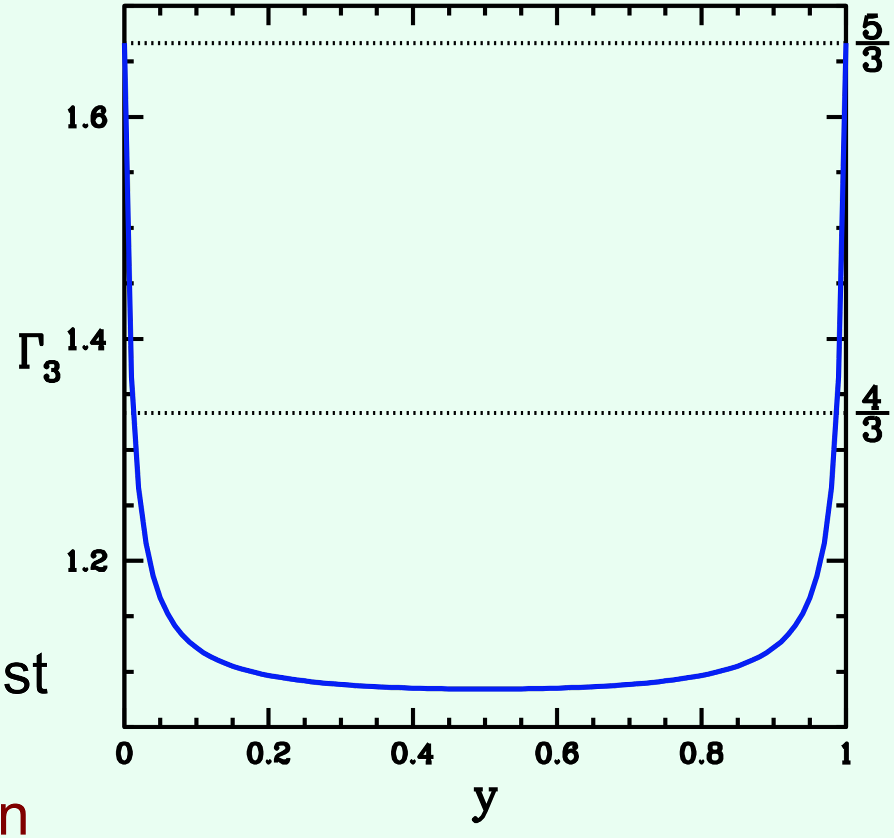
In response to temperature increase



Adiabatic Exponents: Ionization Zones

$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho} \right)_{\text{ad}}$$

Gas is compressed
→ increase in temperature
→ rapid increase in ionization
→ energy goes into ionizing gas
instead of thermal energy
→ temperature does not rise as fast



Temperature increases slower than normal in response to compression

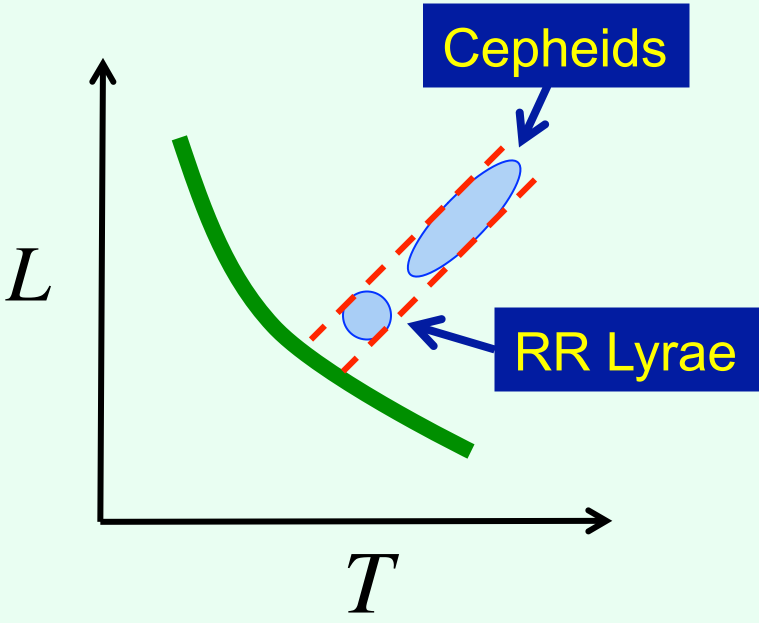
Adiabatic Exponents: Ionization Zones

Adiabatic exponents can be less than 4/3 → unstable!

Ionization zones are responsible for stellar pulsations

- second ionization of *He* provides the most significant driving
- ionization zone must be in the radiative region

Cool, luminous stars (T_{eff}<6300K) will pulsate in an “instability strip” a few hundred K wide on HR diagram



Polytropes

First two structure equations:

$$\boxed{\frac{dm}{dr} = 4\pi r^2 \rho}$$

$$\boxed{\frac{dP}{dr} = -\rho \frac{Gm}{r^2}}$$

Temperature *T* does not appear in these equations explicitly, but is involved implicitly because pressure *P* usually depends on temperature via the equation of state.

In some cases, however, *P* only depends on density and, not, *T*. in these cases, the above two equations are sufficient to define a solution.

A degenerate gas is such a case.

Polytropes

Assume a "polytropic relation" holds throughout the star:

$$\boxed{P = K\rho^{1+\frac{1}{n}}}$$

$K:$ constant

$n:$ polytropic index

Polytropes are useful in two situations:

- The equation of state is really polytropic

  Completely degenerate gas

  - Non-relativistic $P \sim \rho^{5/3}$ $\rightarrow n = 3/2$
  - Ultra-relativistic $P \sim \rho^{4/3}$ $\rightarrow n = 3$

- The equation of state + an additional constraint yields a polytropic relation.
  - Isothermal ideal gas $T = T_0, \quad P \sim \rho T \sim \rho$ $\rightarrow n = \infty$
  - Fully convective star: convection maintains a fixed *T* gradient

$$T \sim P^{2/5}, \quad P \sim \rho T \sim \rho P^{2/5} \rightarrow P \sim \rho^{5/3} \qquad \rightarrow n = 3/2$$

Polytropes: Lane-Emden Equation

Hydrostatic equilibrium

$$\boxed{\frac{dP}{dr}=-\rho\frac{Gm}{r^2}} \quad \rightarrow \frac{r^2}{\rho}\frac{dP}{dr}=-Gm \quad \rightarrow \frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right]=-G\frac{dm}{dr}$$

$$\rightarrow \frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right]=-G4\pi r^2\rho \qquad \rightarrow \frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right]=-4\pi G\rho$$

Make this equation dimensionless. First density,

$$\boxed{\rho(r)=\rho_c\theta^n(r)} \qquad \rho_c=\rho(r=0)$$

$$P=K\rho^{1+\frac{1}{n}} \quad \rightarrow P=K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \quad =P_c\theta^{n+1} \qquad \left(P_c=K\rho_c^{1+\frac{1}{n}}\right)$$

Polytropes: Lane-Emden Equation

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right]=-4\pi G\rho \qquad \rho=\rho_c\theta^n \qquad P=P_c\theta^{n+1}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho_c\theta^n}\frac{d}{dr}\left(P_c\theta^{n+1}\right)\right]=-4\pi G\rho_c\theta^n$$

$$\rightarrow\frac{P_c}{\rho_c}\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\theta^n}(n+1)\theta^n\frac{d\theta}{dr}\right]=-4\pi G\rho_c\theta^n$$

$$\rightarrow\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta}{dr}\right]=-\theta^n$$

Next, make radius dimensionless: $\boxed{\xi=\frac{r}{r_n}}$ $\boxed{r_n=\left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}}$

Polytropes: Lane-Emden Equation

$$r_n^2 \frac{1}{(\xi r_n)^2} \frac{d}{d(\xi r_n)} \left[(\xi r_n)^2 \frac{d\theta}{d(\xi r_n)} \right] = -\theta^n$$

$$\rightarrow \boxed{\frac{1}{\xi^2} \frac{d}{d\xi} \left[\xi^2 \frac{d\theta}{d\xi} \right] = -\theta^n}$$

Lane-Emden equation

$$\rightarrow \frac{1}{\xi^2} \left(2\xi \frac{d\theta}{d\xi} + \xi^2 \frac{d^2\theta}{d\xi^2} \right) = -\theta^n$$

$$\rightarrow \frac{d^2\theta}{d\xi^2} + \frac{2}{\xi} \frac{d\theta}{d\xi} + \theta^n = 0$$

$$\boxed{\theta'' = -\frac{2}{\xi}\theta' - \theta^n}$$

Polytropes: Lane-Emden Equation

$$\boxed{\theta'' = -\frac{2}{\xi}\theta' - \theta^n}$$

$$\boxed{\xi = \frac{r}{r_n}}$$

$$\boxed{\rho(r) = \rho_c \theta^n(r)}$$

Boundary conditions

- Center

$$r = 0 \rightarrow \boxed{\xi = 0} \qquad \rho = \rho_c \rightarrow \boxed{\theta = 1} \qquad \frac{d\rho}{dr} = 0 \rightarrow \boxed{\theta' = 0}$$

- Surface is at first zero crossing of $\theta(\xi)$

$$\rho = 0 \rightarrow \boxed{\theta = 0} \qquad \xi = \xi_1$$

Solving Lane-Emden

The Lane-Emden equation can be integrated numerically outward until $\theta = 0$.

- Start at $\xi = 0$. In steps of $d\xi$ compute θ'', θ', θ. Stop when $\theta = 0$.
- Get value of ξ_1 , as well as full $\theta(\xi)$ profile.

Polytropes: Lane-Emden Equation

- Analytic solutions exist for three cases: *n*=0, 1, and 5

$$n = 0: \quad \theta(\xi) = 1 - \frac{\xi^2}{6} \qquad \xi_1 = \sqrt{6}$$

$$n = 1: \quad \theta(\xi) = \frac{\sin\xi}{\xi} \qquad \xi_1 = \pi$$

$$n = 5: \quad \theta(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \qquad \xi_1 = \infty$$

- Polytropes with *n*>5 have infinite (divergent) mass.

- Only models with *n*=3/2 and *n*=3 are physically relevant.

$$n = \frac{3}{2}: \quad P = K\rho^{5/3} \qquad n = 3: \quad P = K\rho^{4/3}$$

Polytropes: Lane-Emden Equation

Solving a differential equation numerically

Simplest case: $\frac{dy}{dx} = f(x,y)$ Boundary condition: $y(0) = y_0$

$$y'(0) = y'_0$$

- Choose step Δx

- Start with boundary condition and evaluate y at next step

$$y(\Delta x) = y(0) + y'(0) \cdot \Delta x$$

- Repeat for all subsequent steps

$$y(x) = y(x - \Delta x) + \frac{dy}{dx}(x - \Delta x) \cdot \Delta x$$

Polytropes: Lane-Emden Equation

Solving a differential equation numerically

- Loop over steps:

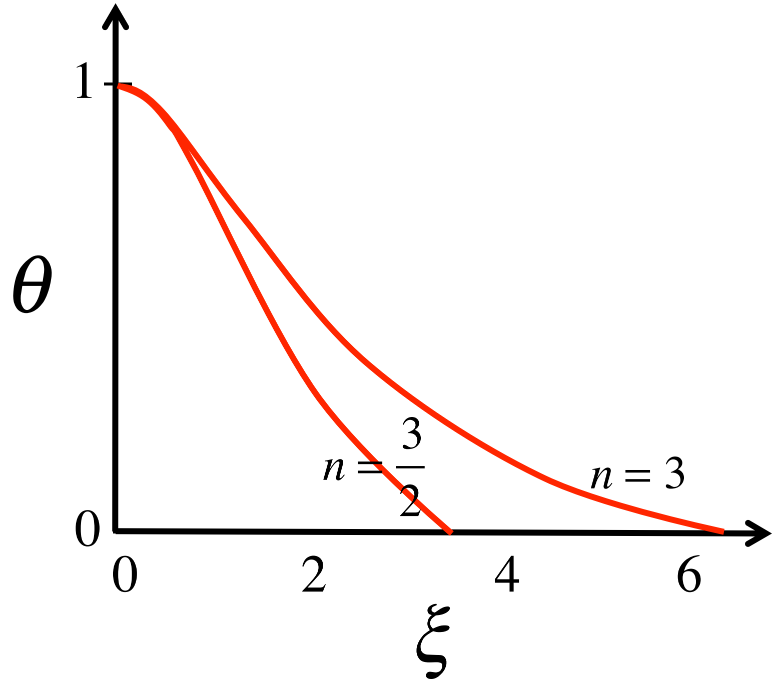
$$y[i] = y[i-1] + \frac{dy}{dx}[i-1] \cdot \Delta x$$

$$\frac{dy}{dx}[i] = \frac{dy}{dx}[i-1] + \frac{d^2y}{dx^2}[i-1] \cdot \Delta x$$

$$\frac{d^2y}{dx^2}[i] = f(x, y, y')$$

Polytropes: Lane-Emden Equation

- As *n* increases, solutions become less centrally concentrated



Polytropes

- Total mass enclosed by *r* $$M(<r)=\int_0^r \rho(r)4\pi r^2\,dr$$

$$M(<\xi)=\int_0^r \rho_c\theta^n 4\pi(\xi r_n)^2\,d(\xi r_n) \quad =4\pi r_n^3\rho_c\int_0^r \theta^n\xi^2\,d\xi$$

$$=4\pi r_n^3\rho_c\int_0^r \xi^2\left[-\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)\right]d\xi \qquad =4\pi r_n^3\rho_c\left[-\xi^2\frac{d\theta}{d\xi}\right]$$

- Total mass in star $$\boxed{M=4\pi r_n^3\rho_c\left[-\xi_1^2\theta'(\xi_1)\right]}$$

- Total radius of star $$\boxed{R=r_n\xi_1=\left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}\xi_1}$$

- Pressure - density $$\boxed{P_c=K\rho_c^{1+\frac{1}{n}}}$$

Polytropes

We have four equations → we can get 4 unknowns.

e.g., if we know the equation of state: *K*, *n*, and the mass *M*, we can

- solve Lane-Emden to get ξ_1 and $\theta'\left(\xi_1\right)$

$$\boxed{\theta'' = -\frac{2}{\xi}\theta' - \theta^n}$$

- use four equations to get $r_n,\ R,\ \rho_c,\ P_c$

$$\boxed{P_c = K\rho_c^{1+\frac{1}{n}}}$$

$$\boxed{R = r_n\xi_1 = \left[\frac{\left(n+1\right)P_c}{4\pi G\rho_c^2}\right]^{1/2}\xi_1}$$

$$\boxed{M = 4\pi r_n^3\rho_c\left[-\xi_1^2\theta'\left(\xi_1\right)\right]}$$

Polytropes: Structure of a White Dwarf

$$\boxed{P_c = K\rho_c^{1+\frac{1}{n}}}$$

$$\boxed{M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right]}$$

$$\boxed{R = r_n \xi_1 = \left[\frac{(n+1)P_c}{4\pi G \rho_c^2}\right]^{1/2} \xi_1}$$

Consider a White Dwarf composed of a degenerate, non-relativistic electron gas.

$$n = \frac{3}{2}, \quad P = K\rho^{5/3}$$

$$R^2 \sim \frac{P_c}{\rho_c^2} \sim \frac{\rho_c^{5/3}}{\rho_c^2} = \rho_c^{-1/3} \rightarrow \rho_c \sim R^{-6}$$

$$M \sim r_n^3 \rho_c \sim R^3 R^{-6} \sim R^{-3} \rightarrow \boxed{R \sim M^{-1/3}}$$

The radius of a White Dwarf shrinks as its mass increases!

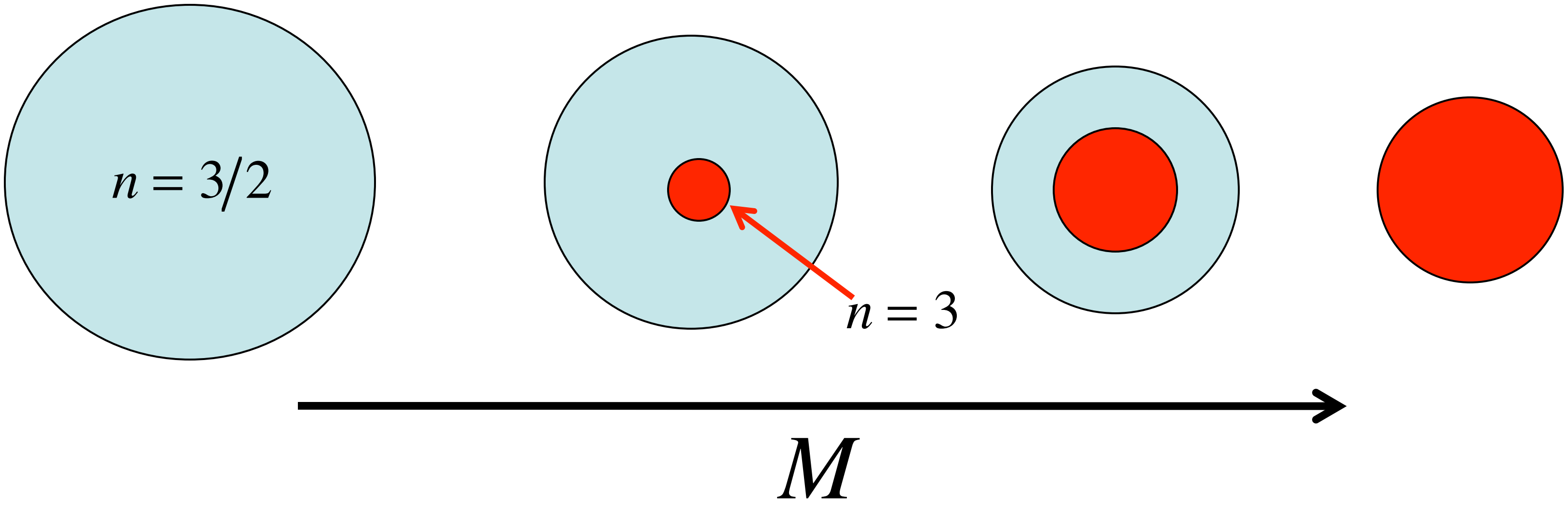
Polytropes: Structure of a White Dwarf

$$\boxed{R \sim M^{-1/3}} \qquad \boxed{\rho_c \sim R^{-6}}$$

As the white dwarf mass increases, its radius shrinks and its central density increases.

Eventually, the core will become relativistic.

As the mass keeps growing, the relativistic core grows too.



Polytropes: Structure of a White Dwarf

$$\boxed{P_c = K\rho_c^{1+\frac{1}{n}}}$$

$$\boxed{M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right]}$$

$$\boxed{R = r_n \xi_1 = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2} \xi_1}$$

Consider a White Dwarf composed of a degenerate, ultra-relativistic electron gas. $n = 3, \quad P = K\rho^{4/3}$

$$R^2 \sim \frac{P_c}{\rho_c^2} \sim \frac{\rho_c^{4/3}}{\rho_c^2} = \rho_c^{-2/3} \rightarrow \rho_c \sim R^{-3}$$

$$M \sim r_n^3 \rho_c \sim R^3 R^{-3} \rightarrow \boxed{M = \text{const}}$$

When the white dwarf is fully relativistic, its mass decouples from its radius and central density!

Polytropes: Structure of a White Dwarf

$$\boxed{P_c = K\rho_c^{1+\frac{1}{n}}}$$

$$\boxed{M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right]}$$

$$\boxed{R = r_n \xi_1 = \left[\frac{(n+1)P_c}{4\pi G \rho_c^2}\right]^{1/2} \xi_1}$$

$$r_n = \left(\frac{4P_c}{4\pi G \rho_c^2}\right)^{1/2} = \left(\frac{K\rho_c^{4/3}}{\pi G \rho_c^2}\right)^{1/2} = \left(\frac{K}{\pi G}\rho_c^{-2/3}\right)^{1/2} = \left(\frac{K}{\pi G}\right)^{1/2} \rho_c^{-1/3}$$

$$M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right] = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} \rho_c^{-1} \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right]$$

$$\boxed{M = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} \left[-\xi_1^2 \theta'(\xi_1)\right]}$$

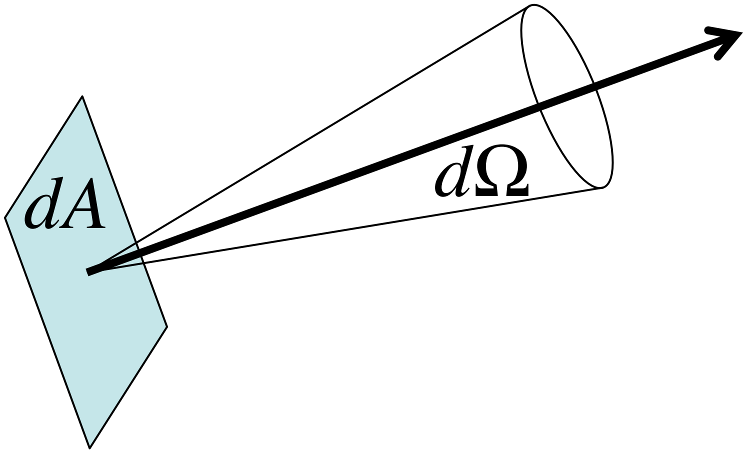
Plugging in numbers:

$$\boxed{M = 1.46 M_\odot}$$

This is the Chandrasekhar limiting mass for white dwarfs.
No white dwarfs are observed with mass greater than this.

Radiative Transfer

Consider energy in the form of light, passing through a medium. The amount of energy passing through depends on location, direction of flow, time, frequency of light, area through which light is passing.



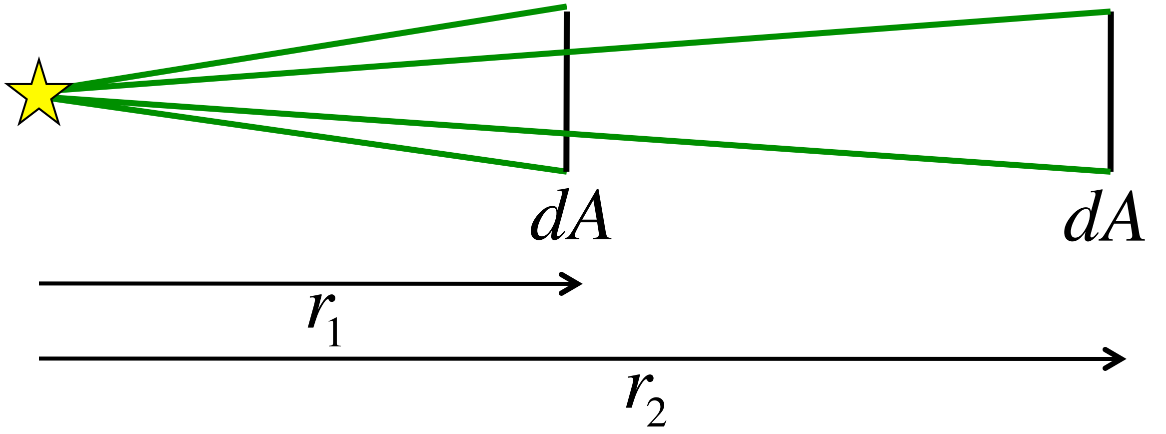
Radiative Transfer

E	erg	Energy
E_ν	$\mathrm{erg\ Hz^{-1}}$	Specific Energy
L	$\mathrm{erg\ s^{-1}}$	Luminosity
L_ν	$\mathrm{erg\ s^{-1}Hz^{-1}}$	Specific Luminosity
F	$\mathrm{erg\ s^{-1}cm^{-2}}$	Flux
F_ν	$\mathrm{erg\ s^{-1}cm^{-2}Hz^{-1}}$	Specific Flux
I	$\mathrm{erg\ s^{-1}cm^{-2}st^{-1}}$	Intensity
I_ν	$\mathrm{erg\ s^{-1}cm^{-2}st^{-1}Hz^{-1}}$	Specific Intensity

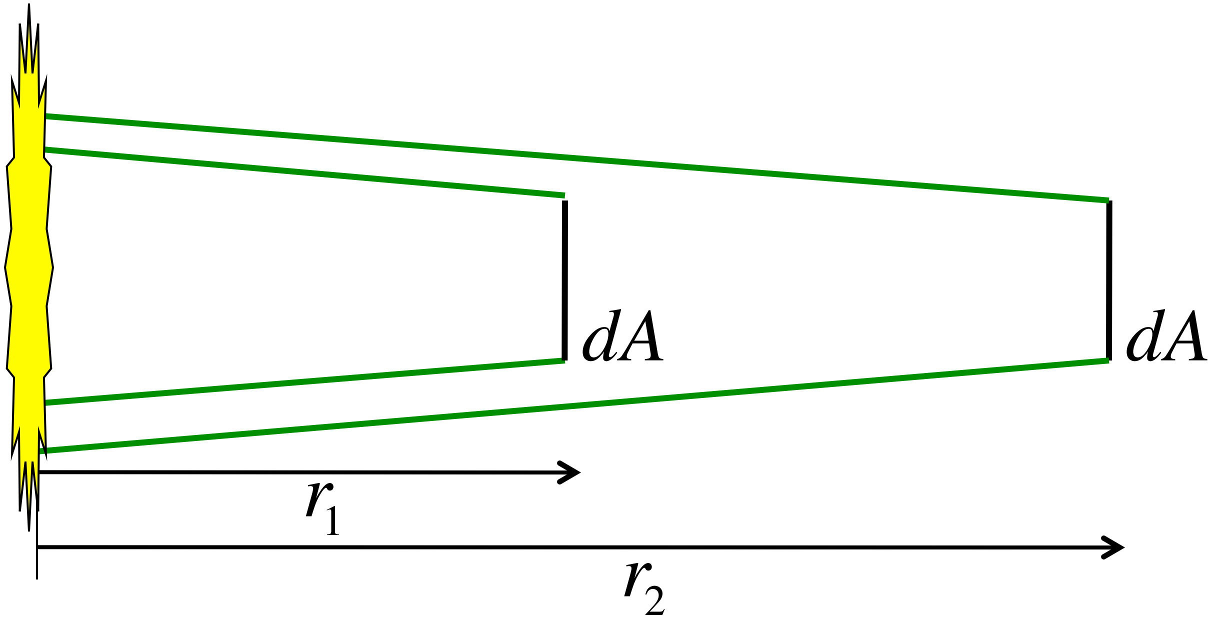
$$\boxed{dE = I_\nu \cdot dt \cdot dA \cdot d\Omega \cdot d\nu}$$

Radiative Transfer

I_ν is constant along a ray travelling on a path, whereas flux decreases as r^{-2}.



Fewer rays from a given source cross an area dA that is farther away because that area is a smaller fraction of the whole sphere at that radius.



However, an equal number of rays cross the area dA per solid angle because as dA is further away, a fixed solid angle corresponds to a larger area of emitting region by ~r^2.

Radiative Transfer: Emission

Electrons can recombine with atoms or drop to lower energy levels and emit photons

Emission coefficient

$$j \quad \mathrm{erg\ s^{-1}cm^{-2}st^{-1}cm^{-1}}$$

$$j_\nu \quad \mathrm{erg\ s^{-1}cm^{-2}st^{-1}Hz^{-1}cm^{-1}}$$

$$dE = j_\nu \cdot dt \cdot dV \cdot d\Omega \cdot d\nu$$

$$\boxed{dI_\nu = j_\nu ds}$$

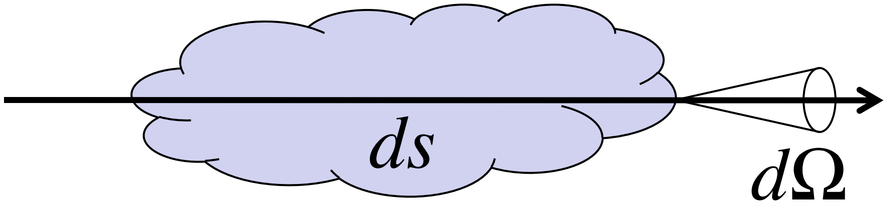
j_ν is the change of I_ν per unit length along a path.

We are only considering spontaneous emission here. Stimulated emission depends on I_ν and is more conveniently treated as a "negative absorption".

Radiative Transfer: Absorption

Absorption

As light passes through a medium, it can be absorbed.



$$\boxed{dI_\nu = -\alpha_\nu I_\nu ds}$$

$\alpha_\nu ds$: fractional loss of intensity over path *ds*

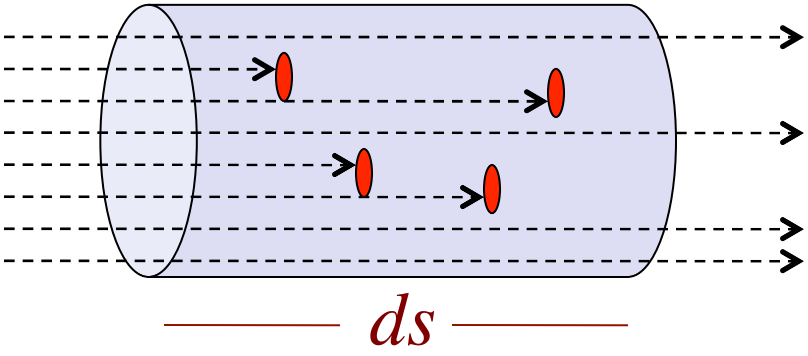
α_ν : cm^{-1}

positive $\alpha_\nu \rightarrow$ energy loss

The change in intensity depends on intensity itself because the more photons try to move through a medium, the more photons can become absorbed by it.

Radiative Transfer: Absorption

If absorption is due to particles with number density n and cross-sectional area σ_ν



Consider a volume element

$$ds \cdot dA$$

Total area covered by particles:

$$n \cdot dV \cdot \sigma_\nu \quad = n \cdot dA \cdot ds \cdot \sigma_\nu$$

Fraction of area that is covered by particles:

$$\frac{n \cdot dA \cdot ds \cdot \sigma_\nu}{dA} = n \cdot \sigma_\nu \cdot ds \quad = \alpha_\nu ds \quad \rightarrow \quad \boxed{\alpha_\nu = n\sigma_\nu}$$

Define opacity as: $\boxed{\alpha_\nu = \rho\kappa_\nu}$ $\qquad \kappa_\nu : \mathrm{cm}^2\mathrm{g}^{-1}$

Opacity is a property of the intervening material, like a cross-sectional area per gram of material.

Radiative Transfer: Optical Depth

Define optical depth τ_ν

$$\boxed{d\tau_\nu = \alpha_\nu ds}$$

$$dI_\nu = -\alpha_\nu I_\nu ds \quad \rightarrow \frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \quad \rightarrow I_\nu = I_\nu(0)e^{-\alpha_\nu s}$$

$$dI_\nu = -I_\nu d\tau_\nu \quad \rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu \quad \rightarrow I_\nu = I_\nu(0)e^{-\tau_\nu}$$

Optical depth is the number of e-foldings change in intensity.
It is an alternative variable for path.

- Fixed ds = fixed distance
- Fixed $d\tau$ = fixed change of I_ν

Radiative Transfer Equation

Change in intensity over a path length is the sum of emission + absorption (source + sink terms).

$$dI_\nu = -\alpha_\nu I_\nu ds + j_\nu ds \qquad \rightarrow \qquad \boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu}$$

$$dI_\nu = -I_\nu d\tau_\nu + j_\nu \frac{d\tau_\nu}{\alpha_\nu} \qquad \rightarrow \qquad \boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu}$$

$$\boxed{S_\nu = \frac{j_\nu}{\alpha_\nu}}$$ Source function

In stars, the source function is close to the blackbody Planck function.

Radiative Transfer: Net Flux

- If area *dA* is perpendicular to light rays

$$dF_\nu = I_\nu d\Omega$$

- If normal vector to area *dA* is at an angle θ to light rays

$$dF_\nu = I_\nu \cos\theta d\Omega$$

Flux is reduced because effective area is smaller.

- "Net flux" in the direction of $\vec{n}$ is $\boxed{F_\nu(\vec{n}) = \int I_\nu \cos\theta \, d\Omega}$

For an isotropic radiation field intensity is angle-independent.

$$F_\nu(\vec{n}) = I_\nu \int \cos\theta \, d\Omega = 0$$

Radiative Transfer Equation

$$\frac{dI_\nu}{d\tau_\nu} + I_\nu = S_\nu \qquad \rightarrow \; e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\rightarrow \frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right) = e^{\tau_\nu} S_\nu \qquad \rightarrow \; I_\nu e^{\tau_\nu} = \int_0^{\tau_\nu} e^{\tau'_\nu} S_\nu d\tau'_\nu \; + \; \text{const}$$

When $\tau_\nu = 0 \;\rightarrow\; I_\nu = I_\nu(0) \;\rightarrow\; \text{const} = I_\nu(0)$

$$\rightarrow \; I_\nu e^{\tau_\nu} = I_\nu(0) + \int_0^{\tau_\nu} e^{\tau'_\nu} S_\nu d\tau'_\nu$$

$$\rightarrow \; I_\nu = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu d\tau'_\nu$$

Radiative Transfer Equation

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu d\tau'_\nu$$

Final intensity is initial intensity diminished by absorption
+ integrated source function also diminished by absorption.

For $S_\nu = \text{const:}$ $I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu$

$$= I_\nu(0)e^{-\tau_\nu} + S_\nu \left[e^{-(\tau_\nu - \tau'_\nu)} \right]_0^{\tau_\nu}$$

$$\boxed{I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)}$$

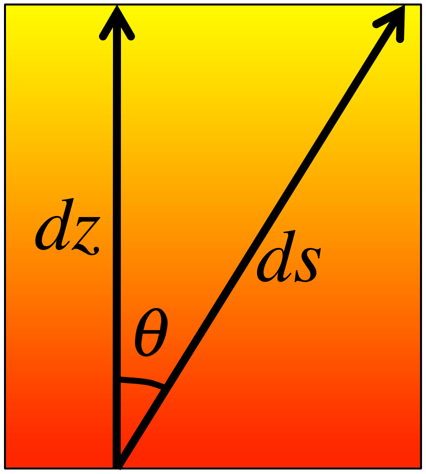
as $\tau_\nu \to \infty, \quad I_\nu \to S_\nu$

Radiative Transfer: Radiative Diffusion

Within a star there is a net radial flux (outward) causing a departure from isotropy. We can relate this flux to the temperature gradient.

- plane-parallel approximation
- angular dependence through θ only
- $\mu = \cos\theta$

$$\boxed{ds = dz/\mu}$$



$$\frac{dI_\nu}{ds} = -\alpha_\nu\left(I_\nu - S_\nu\right) \quad = -\rho\kappa_\nu\left(I_\nu - S_\nu\right)$$

$$\rightarrow \mu\frac{\partial I_\nu}{\partial z} = -\rho\kappa_\nu\left(I_\nu - S_\nu\right) \qquad \rightarrow I_\nu = S_\nu - \frac{\mu}{\rho\kappa_\nu}\frac{\partial I_\nu}{\partial z}$$

Radiative Transfer: Radiative Diffusion

$$\boxed{I_\nu = S_\nu - \underbrace{\frac{\mu}{\rho\kappa_\nu}\frac{\partial I_\nu}{\partial z}}_{\big\uparrow}}$$

$\rho\kappa_\nu$ is the fractional loss of I_ν per unit length.

$\left(\rho\kappa_\nu\right)^{-1}$ is the mean-free-path.

This second term is the change in intensity over the mean-free-path.

This is small $\rightarrow I_\nu \approx S_\nu \approx B_\nu$

To first order: $I_\nu \approx B_\nu - \dfrac{\mu}{\rho\kappa_\nu}\dfrac{\partial B_\nu}{\partial z}$

Integrate to get flux in z-direction: $F_\nu(z) = \int I_\nu \cos\theta \, d\Omega \;\; = -2\pi \int_{+1}^{-1} I_\nu(\mu, z)\,\mu \, d\mu$

$$d\Omega = 2\pi \sin\theta d\theta = -2\pi d\mu$$

Radiative Transfer: Radiative Diffusion

$$\boxed{I_\nu \approx B_\nu - \frac{\mu}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}}$$

$$F_\nu(z) = 2\pi\int_{-1}^{+1} I_\nu(\mu,z)\,\mu\,d\mu$$

B_ν is isotropic so only the derivative term has a non-zero angle dependence.

$$F_\nu(z) = 2\pi\int_{-1}^{+1}\left(-\frac{\mu}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\right)\mu\,d\mu \quad = -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\int_{-1}^{+1}\mu^2\,d\mu$$

$$= -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\left[\frac{\mu^3}{3}\right]_{-1}^{+1} \quad = -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\frac{2}{3}$$

$$\boxed{F_\nu(z) = -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z}}$$

Radiative Transfer: Radiative Diffusion

$$F_\nu(z) = -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z}$$

Total flux (integrating over frequency):

$$F(z) = \int_0^\infty F_\nu(z)\,d\nu \qquad = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu$$

Define the Rosseland mean opacity:

$$\rho\kappa_R \equiv \frac{\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu}$$

Can be computed for any material, at any density and any temperature (albeit with difficulty).

Radiative Transfer: Radiative Diffusion

$$F(z) = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}d\nu$$

$$= -\frac{4\pi}{3}\frac{\partial T}{\partial z}\frac{1}{\rho\kappa_R}\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu$$

$$= -\frac{4\pi}{3}\frac{\partial T}{\partial z}\frac{1}{\rho\kappa_R}\frac{\partial}{\partial T}\int_0^\infty B_\nu\, d\nu$$

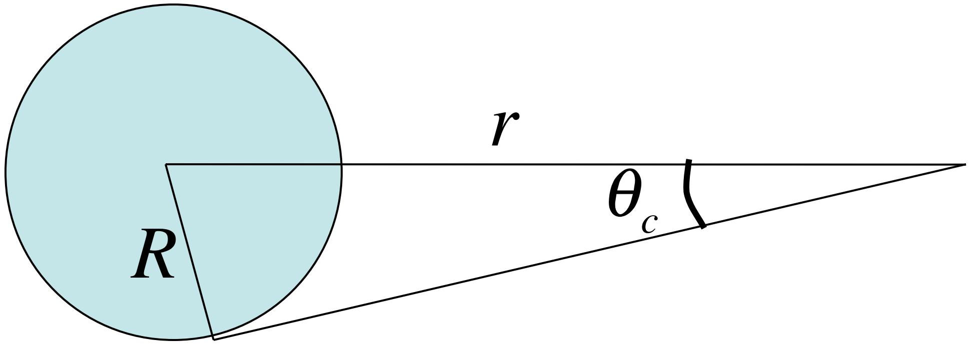
$$= -\frac{4\pi}{3\rho\kappa_R}\frac{\partial T}{\partial z}\frac{\partial B}{\partial T}$$

where *B* is the total intensity

Radiative Transfer: Radiative Diffusion

Sphere of uniform brightness.
At an exterior point,

$$I = \begin{cases} B, & \theta < \theta_c \\ 0, & \theta > \theta_c \end{cases}$$



Find flux: $F = \int I \cos\theta \, d\Omega \;= B \int_0^{2\pi} d\phi \int_0^{\theta_c} \cos\theta \sin\theta \, d\theta$

$$= 2\pi B \int_0^{\theta_c} \cos\theta \sin\theta \, d\theta \;= 2\pi B \int_0^{\sin\theta_c} x \, dx \;= 2\pi B \frac{\sin^2\theta_c}{2}$$

$$\sin\theta_c = \frac{R}{r}$$

$$\boxed{F = \pi B \left(\frac{R}{r}\right)^2}$$

At surface: $\boxed{F = \pi B}$
r=R

Radiative Transfer: Radiative Diffusion

We found that the flux coming from an enclosed sphere is:

Also, for a blackbody:

$$\left.\begin{aligned} F &= \pi B \\ F &= \sigma T^4 \end{aligned}\right\} B = \frac{\sigma T^4}{\pi}$$

$$\boxed{F(z) = -\frac{4\pi}{3\rho\kappa_R}\frac{\partial T}{\partial z}\frac{\partial B}{\partial T}}$$

$$\frac{\partial B}{\partial T} = \frac{4\sigma T^3}{\pi}$$

$$\boxed{F(z) = -\frac{16\sigma T^3}{3\rho\kappa_R}\frac{\partial T}{\partial z}}$$

<u>Equation of radiative diffusion</u>

- Applies provided that quantities change slowly on scale of mean free path.
- Material properties enter through κ_R

Radiative Transfer: Radiative Diffusion

Spherical symmetry: $z \rightarrow r$

$$\left.\begin{aligned} L_r &= F(r)4\pi r^2 \\ \sigma &= \frac{ac}{4} \end{aligned}\right\} L_r = -\frac{16\left(\frac{ac}{4}\right)T^3}{3\rho\kappa_R}\frac{dT}{dr}4\pi r^2$$

$$\boxed{L_r = -\frac{16\pi acr^2T^3}{3\rho\kappa_R}\frac{dT}{dr}}$$

$$\boxed{\frac{dT}{dr} = -\frac{3\rho\kappa_R L_r}{16\pi acr^2T^3}}$$

The Eddington Limit

Photons carry momentum → absorption of photons must lead to a force.

Spherical symmetric source with luminosity *L*

- Energy flux at distance *r* : $\frac{L}{4\pi r^2}$ $\left(\mathrm{erg\ s^{-1}cm^{-2}}\right)$
- Momentum flux : $\frac{L}{4\pi c r^2}$ $\left(\mathrm{erg\ cm^{-3}}\right)$

$$\left(E = pc\right)$$

- Multiply by opacity to get force per unit mass

$$\left(F = \frac{dp}{dt}\right) \qquad F_{\mathrm{rad}} = \frac{\kappa L}{4\pi c r^2} \quad \left(\mathrm{erg\ cm\ g^{-1}}\right)$$

Opacity is the fraction of momentum flux absorbed per unit mass.

The Eddington Limit

- Inward force per mass due to gravity: $F_{\text{grav}} = \dfrac{GM}{r^2}$

Radiation balances gravity when $F_{\text{rad}} = F_{\text{grav}}$

$$\frac{\kappa L}{4\pi c r^2} = \frac{GM}{r^2} \quad \rightarrow \quad L = \frac{4\pi c G M}{\kappa}$$

At greater luminosities, $F_{\text{rad}} > F_{\text{grav}}$ and gas will be blown away.

- Assume opacity is due to Thompson scattering by free electrons

$$\kappa = \frac{\alpha}{\rho} \quad = \frac{n\sigma}{\rho} \quad = \frac{\sigma}{m} \quad = \frac{\sigma_T}{m_H}$$

$$\boxed{L_{\text{edd}} = \frac{4\pi c G M m_H}{\sigma_T}}$$

$$\boxed{L_{\text{edd}} = 3.2 \times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot}$$

The Eddington Limit

$$L_{\text{edd}} = 3.2 \times 10^4 \left(\frac{M}{M_\odot} \right) L_\odot$$

This is the Eddington limit

Assumptions:
- Thompson scattering only

  other opacity sources increase opacity → lower L_{edd}
- spherical symmetry

- Mass-Luminosity relation for very massive stars

$$\left(\frac{L}{L_\odot} \right) = 34.2 \left(\frac{M}{M_\odot} \right)^{2.4} \quad = L_{\text{edd}} \rightarrow M_{\max} \sim 100\, M_\odot$$

Formation of more massive stars cannot be spherically symmetric

Opacity Sources: Electron Scattering

The opacity of a material depends on the composition $\left(X,Y,Z\right)$ the temperature T and the density ρ of gas.

$$\kappa = \kappa_0 \rho^n T^{-s}$$

1. Electron scattering

$$\kappa = \frac{n\sigma}{\rho} \qquad = \frac{n_e \sigma_e}{\rho}$$

In an ionized mixture of *H* and *He*: $n_e = \frac{\rho N_A}{\mu_e}$ $\qquad \mu_e = \frac{2}{1+X}$

$$\kappa_e = \frac{\rho N_A \left(1+X\right)}{2} \frac{\sigma_e}{\rho} \qquad = \frac{\sigma_e N_A \left(1+X\right)}{2}$$

Opacity Sources: Electron Scattering

If electrons are non-degenerate and non-relativistic, their cross-section is equal to the Thompson cross-section.

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \qquad = 0.6652 \times 10^{-24}\,\mathrm{cm}^2$$

$$\kappa_e = \frac{\sigma_T N_A (1+X)}{2}$$

$$\boxed{\kappa_e = 0.2(1+X)\,\mathrm{cm}^2\mathrm{g}^{-1}}$$

Cannot use if

- heavy elements are abundant or gas is partially ionized
- density is high → degenerate
- temperature is high → relativistic

e$^-$ opacity has no frequency, density or temperature dependence

$$\boxed{\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = s = 0}$$

Opacity Sources: Free-Free Absorption

2\. Free-Free Absorption   A free electron cannot absorb a photon because energy and momentum cannot both be conserved. However, the presence of a charged ion near the electron can make this possible.

$$\gamma + e^- + \text{ion} \rightarrow e^- + \text{ion}$$

inverse of Bremsstrahlung

$$\kappa_{ff} \approx 10^{23} \frac{\rho}{\mu_e} \frac{Z_c^2}{\mu_I} T^{-3.5} \text{cm}^2 \text{g}^{-1}$$

Z_c : average nuclear charge

Since free electrons are required, free-free opacity will be negligible for $T < 10^4 \text{K}$ since *H* will not be ionized.

$$\boxed{\kappa_{ff} \approx 4 \times 10^{22} (X+Y)(1+X) \rho T^{-3.5} \text{cm}^2 \text{g}^{-1}}$$

- fully ionized
- no metals

$$\boxed{\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = 1, s = 3.5}$$

Kramers opacity

Opacity Sources: Bound Absorption

3. Bound-Free Absorption

A photon gets absorbed by a bound electron, ionizing it.

$$\boxed{\kappa_{bf} \approx 4 \times 10^{25} Z(1+X)\rho T^{-3.5} \mathrm{cm}^2 \mathrm{g}^{-1}}$$

- $T > 10^4$ K

$$\boxed{\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = 1, s = 3.5}$$

Kramers opacity

---

4. Bound-Bound Absorption

A photon gets absorbed and causes a transition between bound energy levels in an atom.

- Very complex calculation: absorption line profiles, line broadening.
- ~10 times smaller than f-f or b-f

Kramers opacity

Opacity Sources: H$^-$ Absorption

5. H$^-$ Opacity

At low temperature, an extra electron can attach to the H atom.

H$^-$ has an ionization potential of 0.75 eV so it's very easy to ionize if T > a few thousand K.

$$\kappa_{H^-} \approx 2.5 \times 10^{-31} \left(\frac{Z}{0.02} \right) \rho^{1/2} T^9 \mathrm{cm}^2 \mathrm{g}^{-1}$$

$$3000 < T < 6000\mathrm{K}$$

$$10^{-10} < \rho < 10^{-5}\mathrm{g}$$

$$X \sim 0.7, \quad 0.001 < Z < 0.03$$

Relevant to Sun's atmosphere!

$$\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = 0.5, s = -9$$

Opacity Sources: Tabulated Opacities

In practice, stellar models use opacities calculated using detailed physics.

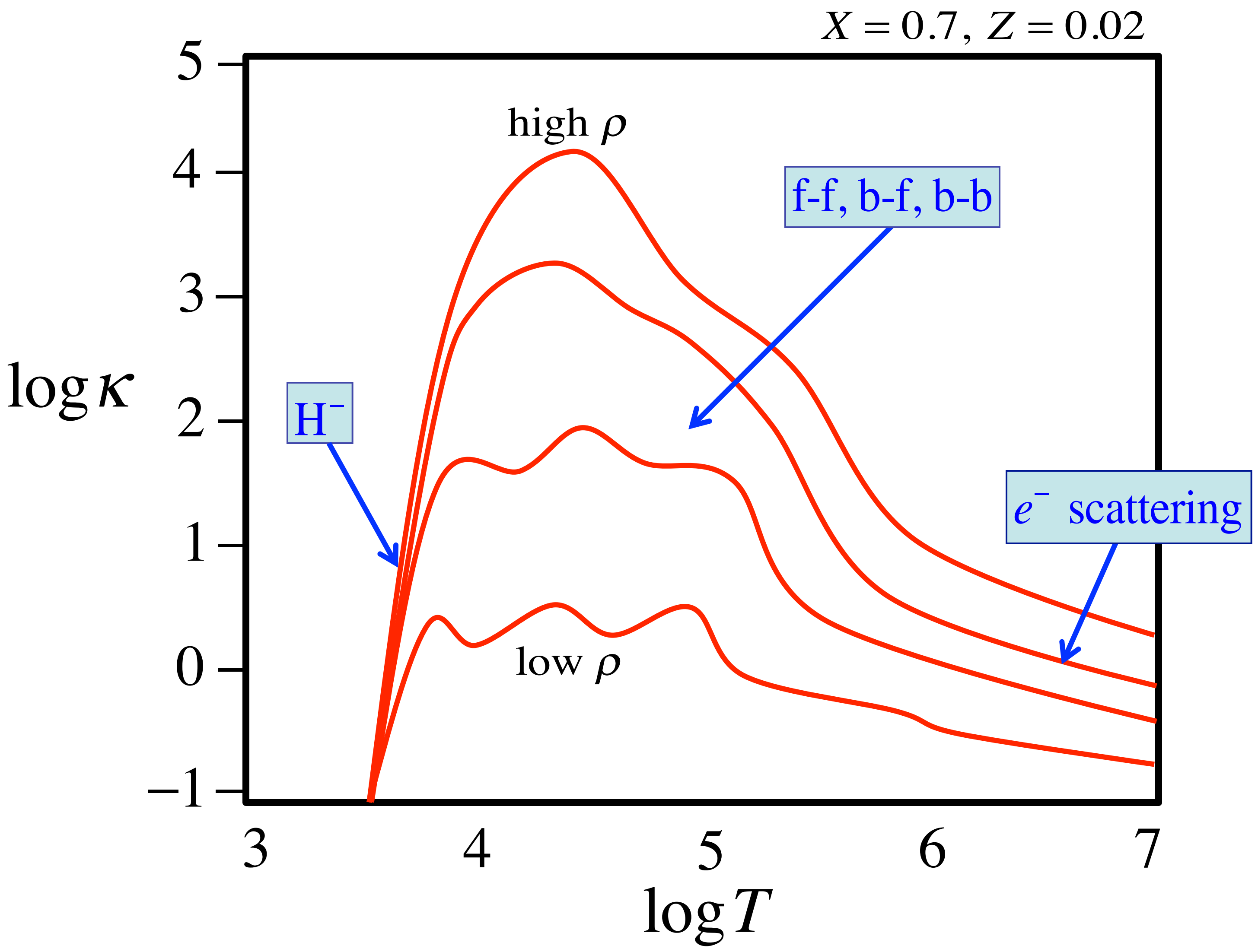
As a function of $\rho, T, X, Y, Z + \text{breakdown of metals}$

- Calculate all the relevant b-b, b-f, f-f, H^-, e^-, scattering effects.
- Get a correct Rosseland mean opacity.

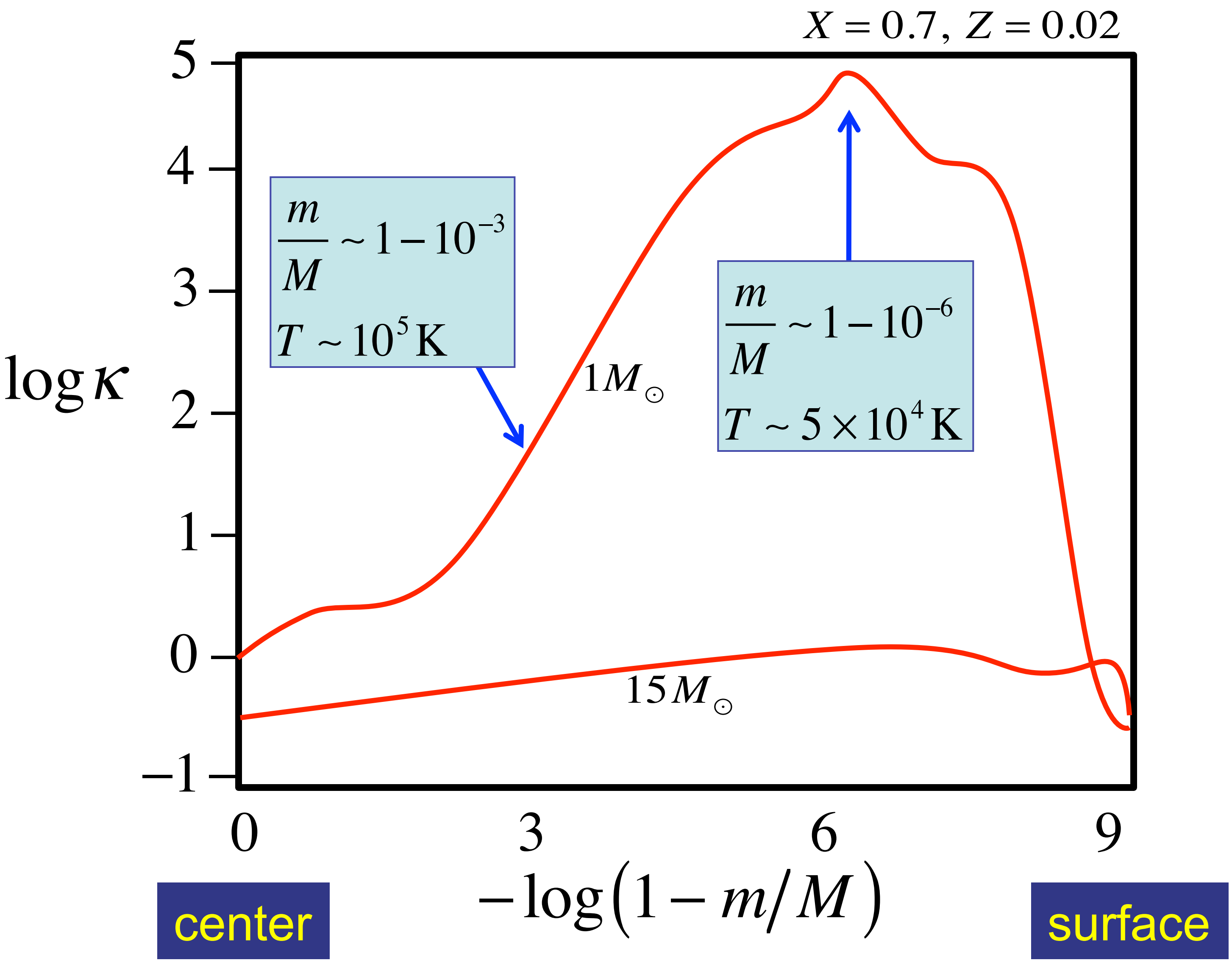
- LANL 1960s following defense calculations
- OPAL Rogers & Iglesias (1992), Iglesias & Rogers (1996)
- Opacity Project Seaton et al. (1994)

- Opacity tables do not cover whole ρ, T, X, Y, Z space. Extrapolation is dangerous.
- Need to do smart interpolation.
- Differences of as much as 30% between projects occur.

Opacity Sources: Tabulated Opacities



Opacity Sources: Tabulated Opacities



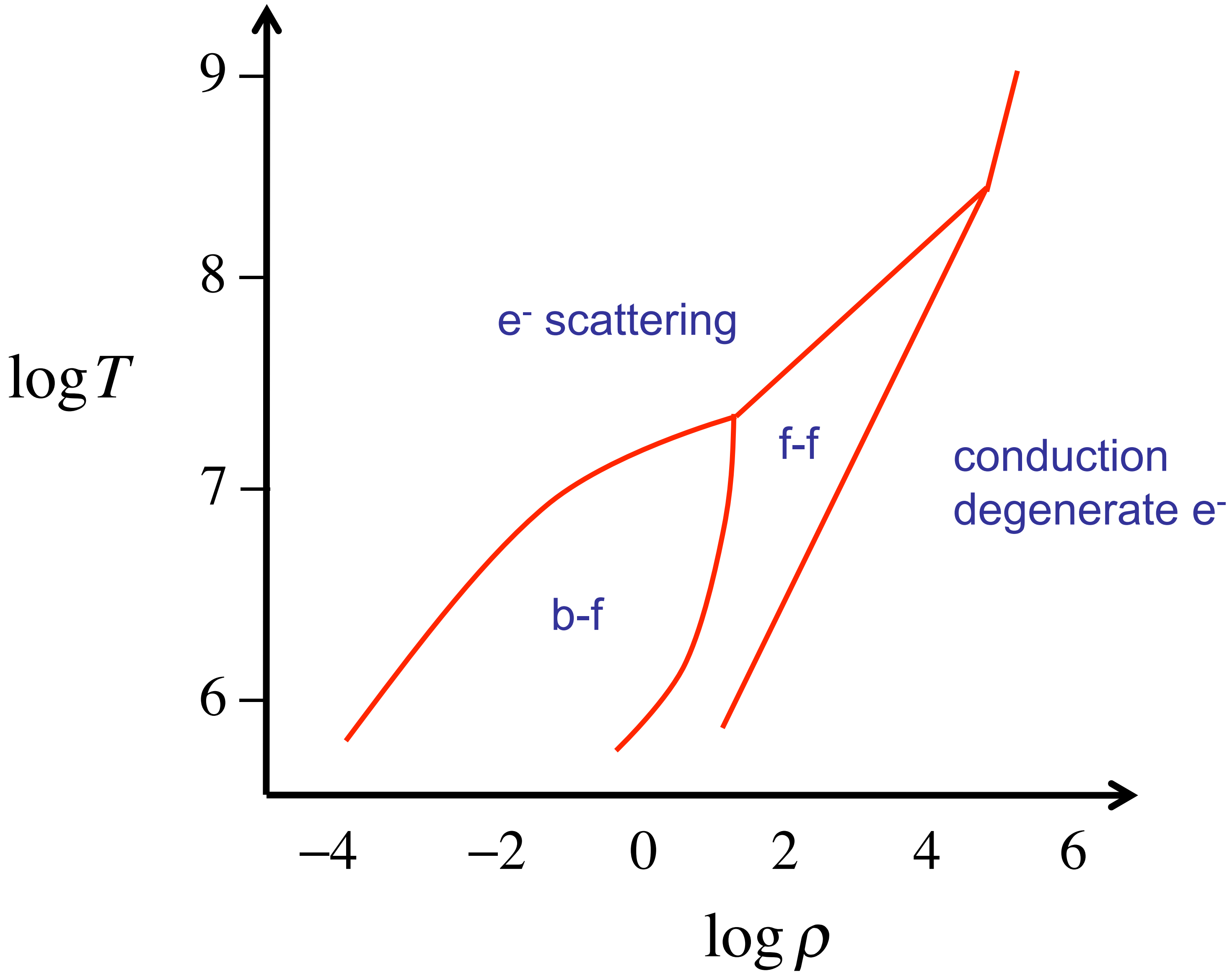
$$\frac{m}{M} \sim 1 - 10^{-3}$$

$$T \sim 10^5\,\mathrm{K}$$

$$\frac{m}{M} \sim 1 - 10^{-6}$$

$$T \sim 5 \times 10^4\,\mathrm{K}$$

Opacity Sources: Tabulated Opacities



Heat Transfer by Conduction

When the density gets very high, degenerate electrons transfer heat by conduction in addition to providing hydrostatic support.

Total energy flux is additive: $F_{\text{tot}} = F_{\text{rad}} + F_{\text{cond}}$

$$F_{\text{rad}} = -\frac{4acT^3}{3\rho\kappa_R}\frac{dT}{dr} \qquad F_{\text{cond}} = -\frac{4acT^3}{3\rho\kappa_{\text{cond}}}\frac{dT}{dr}$$

$$\frac{1}{\kappa_{\text{tot}}} = \frac{1}{\kappa_R} + \frac{1}{\kappa_{\text{cond}}}$$

Like resistance in a parallel circuit

- In normal stars, κ_{cond} is large → conduction is negligible

  In center of sun: $\kappa_R \sim 0.2, \quad \kappa_{\text{cond}} \sim 2\times 10^9$!!!

- In degenerate dense stars, it can be smaller than κ_R

  In center of cool white dwarf: $\kappa_R \sim 0.2, \quad \kappa_{\text{cond}} \sim 5\times 10^{-5}$

Convection

- If luminosity is transported by radiation, then it must obey

$$L_r = -\frac{16\pi a c r^2 T^3}{3\rho\kappa_R}\frac{dT}{dr}$$

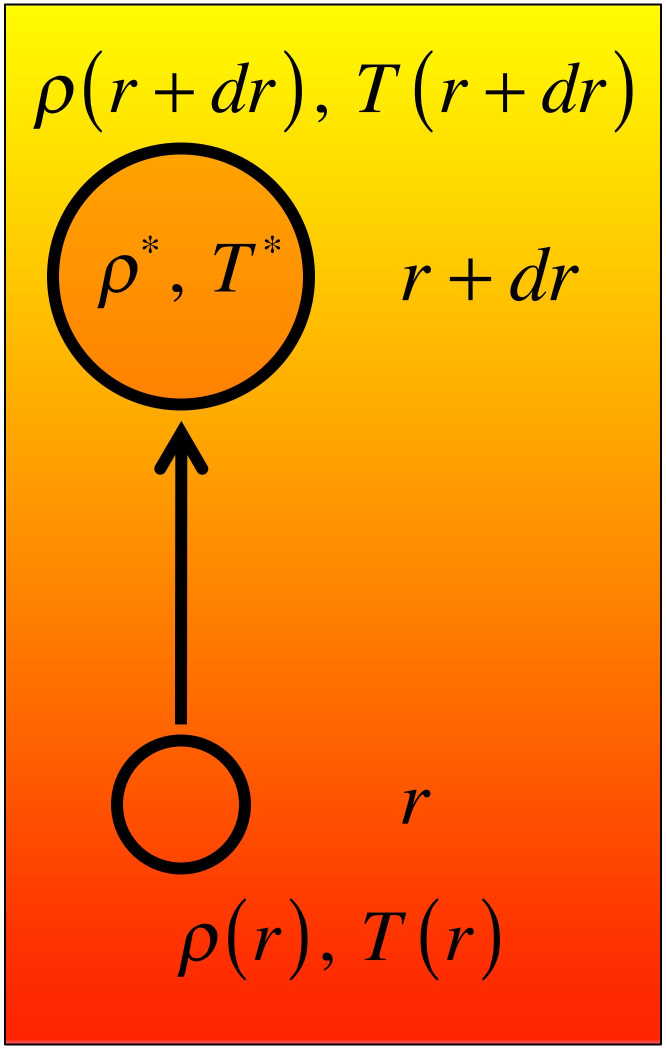
- In a steady state, the energy transported per time at radius *r* must be equal to the energy generation rate in the stellar interior.

$$\rightarrow \frac{dT}{dr} \text{ will be large if } \left\{ \begin{array}{l} L \text{ is large} \\ \kappa_R \text{ is large} \end{array} \right.$$

- The temperature gradient cannot be arbitrarily large. If it gets too steep → convection takes over as the main mode of energy transport.

Convection: Stability Criteria

- Uniform composition, $T(r)$, $\rho(r)$ profiles
- Displace a mass element by *dr* without exchanging heat with the environment (adiabatically).
- Element expands to maintain pressure balance with its environment.
- Its new density ρ^* and temperature T^* will not in general equal the ambient values at $r+dr$.



Convection: Stability Criteria

- Since the change is adiabatic:

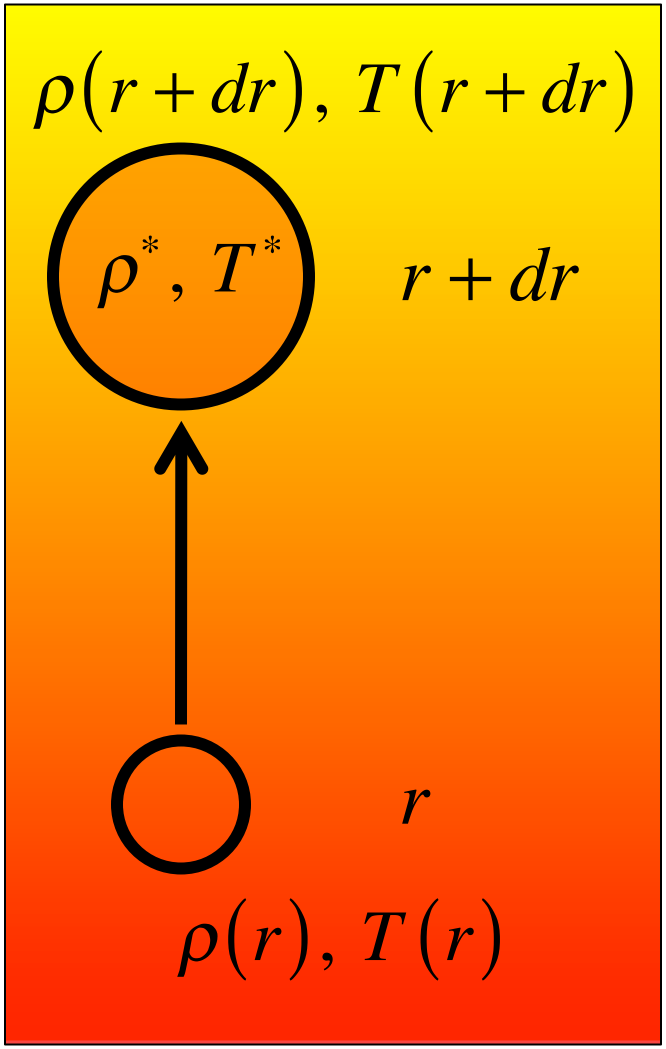
$$\left(\frac{d\ln P}{d\ln\rho}\right)_{\mathrm{ad}} = \Gamma_1 \quad \rightarrow \frac{\rho}{P}\left(\frac{dP}{d\rho}\right)_{\mathrm{ad}} = \Gamma_1$$

$$\rightarrow (d\rho)_{\mathrm{ad}} = \frac{\rho}{\Gamma_1 P}(dP)_{\mathrm{ad}}$$

$$\rho^* = \rho(r) + (d\rho)_{\mathrm{ad}} \quad = \rho(r) + \frac{\rho}{\Gamma_1 P}(dP)_{\mathrm{ad}}$$

$$= \rho(r) + \frac{\rho}{\Gamma_1 P}\left(\frac{dP}{dr}\right)_{\mathrm{ad}} dr$$

- Since pressure equilibrium applies: $\left(\frac{dP}{dr}\right)_{\mathrm{ad}} = \frac{dP}{dr}$



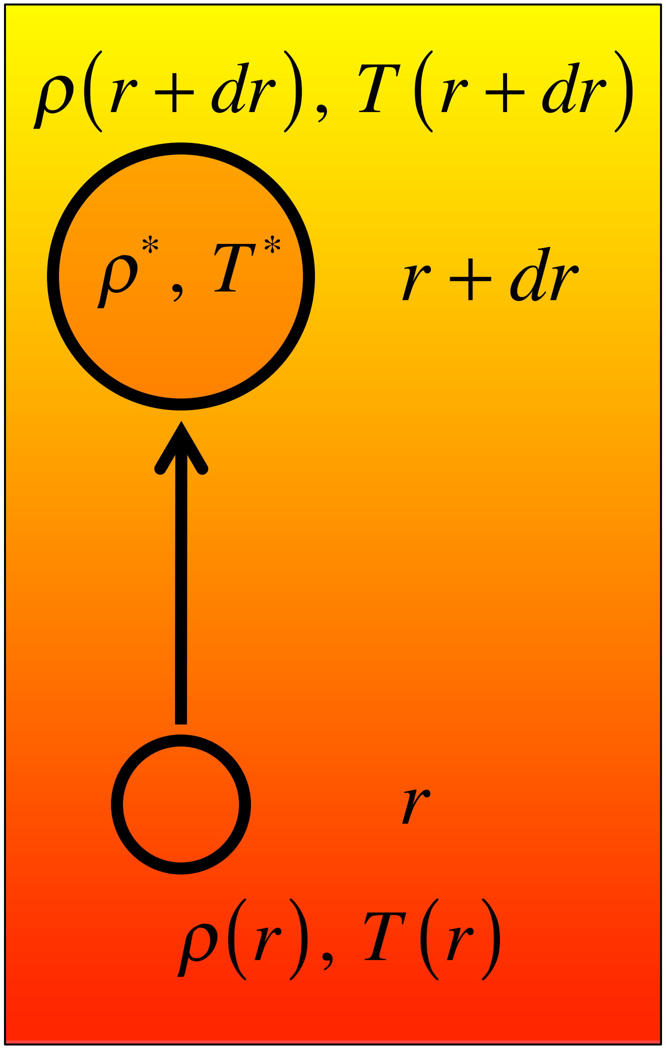
Convection: Stability Criteria

- If $\rho^* > \rho(r+dr)$: displaced element will be denser than its surroundings and will settle back down → STABILITY
- If $\rho^* < \rho(r+dr)$: buoyancy will cause element to rise further → INSTABILITY
- Stability criterion is:

$$\rho(r) + \frac{\rho}{\Gamma_1 P}\frac{dP}{dr}dr > \rho(r+dr)$$

$$\rightarrow \frac{\rho(r+dr) - \rho(r)}{dr} < \frac{\rho}{\Gamma_1 P}\frac{dP}{dr}$$

$$\boxed{\frac{d\rho}{dr} < \frac{\rho}{\Gamma_1 P}\frac{dP}{dr}}$$



Convection: Stability Criteria

- Since the change is adiabatic:

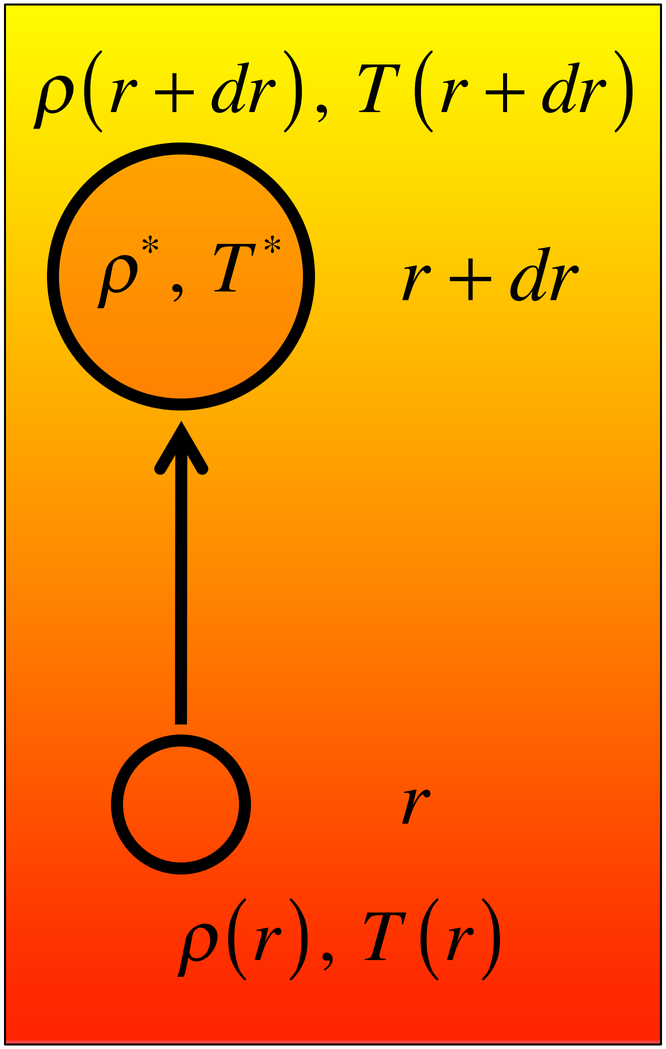
$$\left(\frac{d\ln P}{d\ln T}\right)_{\mathrm{ad}} = \frac{\Gamma_2}{\Gamma_2 - 1} \rightarrow \frac{T}{P}\left(\frac{dP}{dT}\right)_{\mathrm{ad}} = \frac{\Gamma_2}{\Gamma_2 - 1}$$

$$\rightarrow \left(dT\right)_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}\left(dP\right)_{\mathrm{ad}}$$

$$T^* = T\left(r\right) + \left(dT\right)_{\mathrm{ad}}$$

$$= T\left(r\right) + \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\left(dP\right)_{\mathrm{ad}}$$

$$= T\left(r\right) + \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}dr$$



Convection: Stability Criteria

- If $T^* < T(r+dr)$: displaced element will be cooler than its surroundings and will settle back down → STABILITY

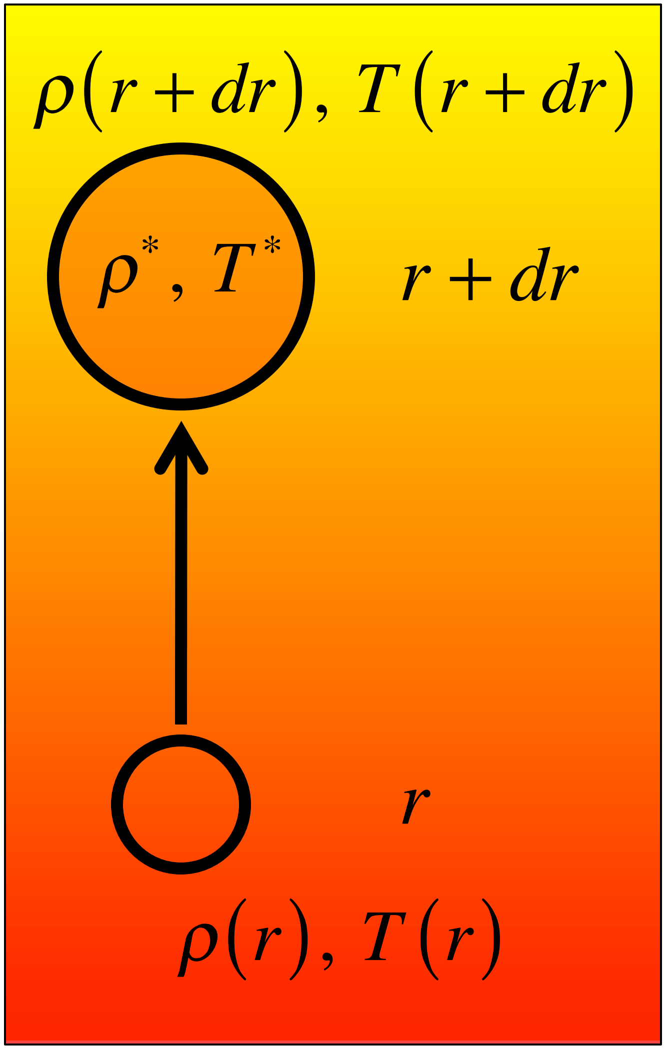
- If $T^* > T(r+dr)$: element will be hotter and thus rise → INSTABILITY

- Stability criterion is:

$$T(r) + \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}dr < T(r+dr)$$

$$\rightarrow \frac{T(r+dr) - T(r)}{dr} > \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\boxed{\frac{dT}{dr} > \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}}$$

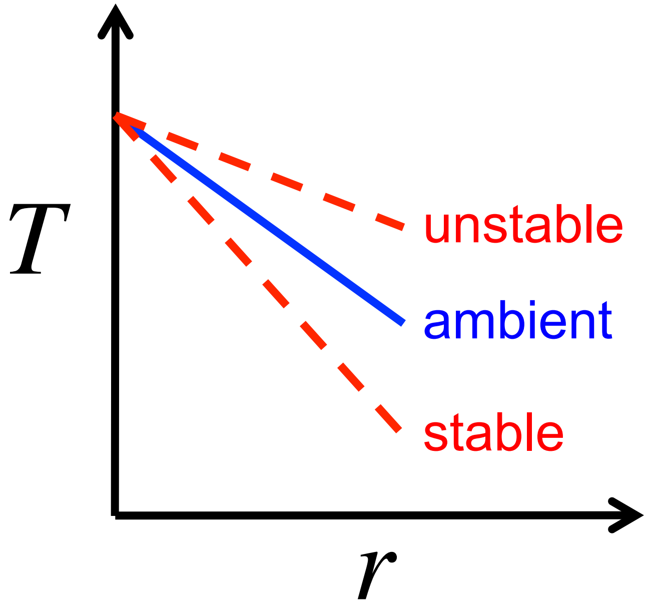


Convection: Stability Criteria

- Stability criterion is:

$$\boxed{\frac{dT}{dr} > \left(\frac{dT}{dr}\right)_{\text{ad}}}$$

$$\boxed{\left|\frac{dT}{dr}\right| < \left|\left(\frac{dT}{dr}\right)_{\text{ad}}\right|}$$



- Too rapid a change in temperature → CONVECTION

$$\boxed{\frac{dT}{dr} > \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}}$$

Convection: Stability Criteria

- Convert to maximum luminosity that can be carried by radiation

$$\frac{dT}{dr} = -\frac{3\rho\kappa_R L_r}{16\pi acr^2T^3} \qquad > \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\frac{3\rho\kappa_R L_r}{16\pi acr^2T^3} < -\left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2}$$

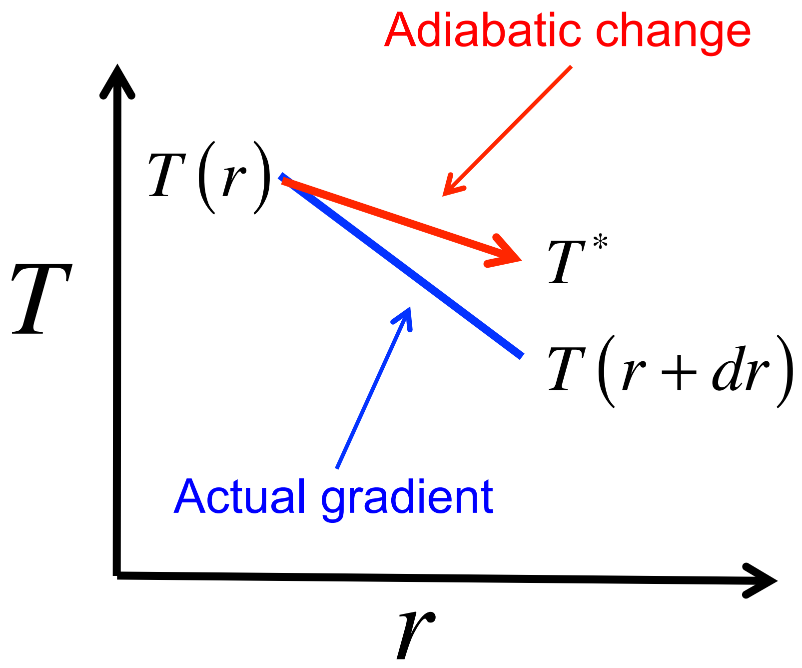
$$\boxed{L_r < -\frac{16\pi acG}{3\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)\frac{T^4m}{P}}$$

- If L exceeds this value → CONVECTION

Convection: Stability Criteria

- In an unstable region, displaced element is hotter than ambient gas
  → continues to rise

- Eventually, radiation will leak out of the rising gas element.
  → extra energy flux from hotter to cooler regions



- What causes convective instability in some regions of a star?

$$L_{\max} \sim \frac{1}{\kappa_R}\left(1-\frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

When $L_{\max}$ is low → convection

Convection: Stability Criteria

Near surface

$$m \approx M$$

$$\frac{T^4}{P} \approx \text{const}$$

$$L_{\max} \sim \frac{1}{\kappa_R}\left(1-\frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

$$\boxed{L_{\max} \sim \frac{1}{\kappa_R}\left(1-\frac{1}{\Gamma_2}\right)}$$

Remember

$$\Gamma_2 = 5/3$$

Vulnerable to convection if

- κ_R is large — Occurs when there is a large contribution from atomic processes.
- $\Gamma_2 \rightarrow 1$ — Occurs in ionization zones where Γ_2 dips below 4/3

So, atomic processes + ionization zones → convection will happen In regions where T~10^4-10^5K and H is being ionized.

Not in high mass stars

Convection: Stability Criteria

Near center

$$L_{\max} \sim \frac{1}{\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

κ_R is low

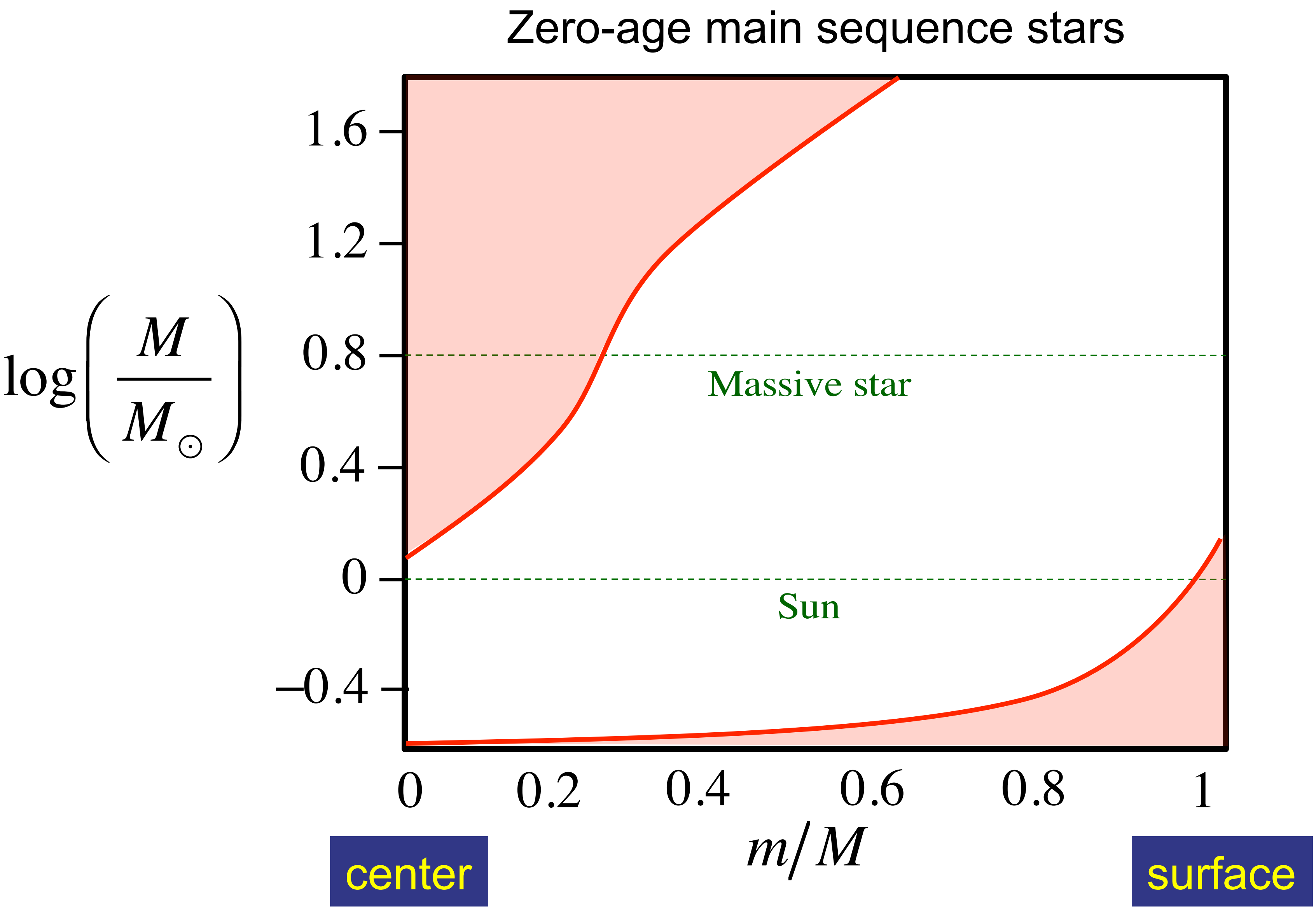
Γ_2 is high

$$\boxed{L_{\max} \sim m}$$

Vulnerable to convection if there is a large luminosity at low mass.

This happens when nuclear energy generation is a very strong function of *T*. CNO in massive stars.

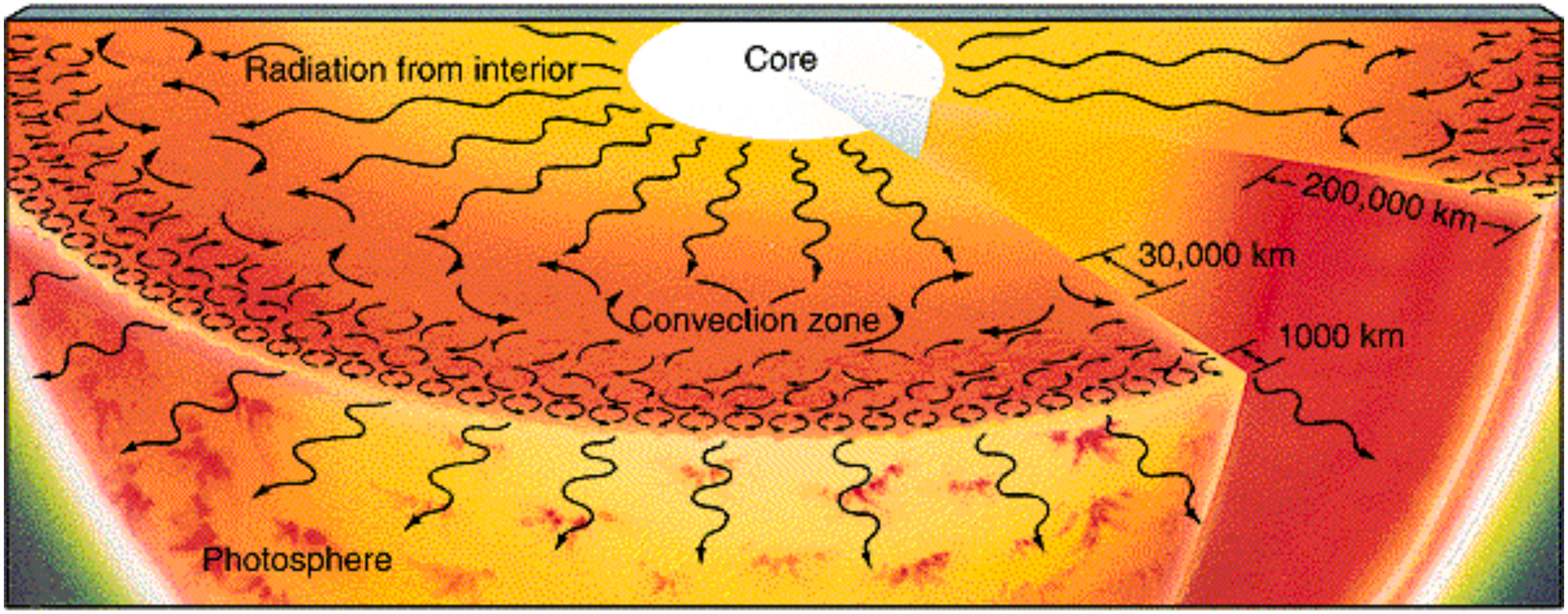
Convection


Zero-age main sequence stars
$\log\left(\frac{M}{M_\odot}\right)$
1.6
1.2
0.8
0.4
0
–0.4
Massive star
Sun
0
0.2
0.4
0.6
0.8
1
m/M
center
surface

Convection

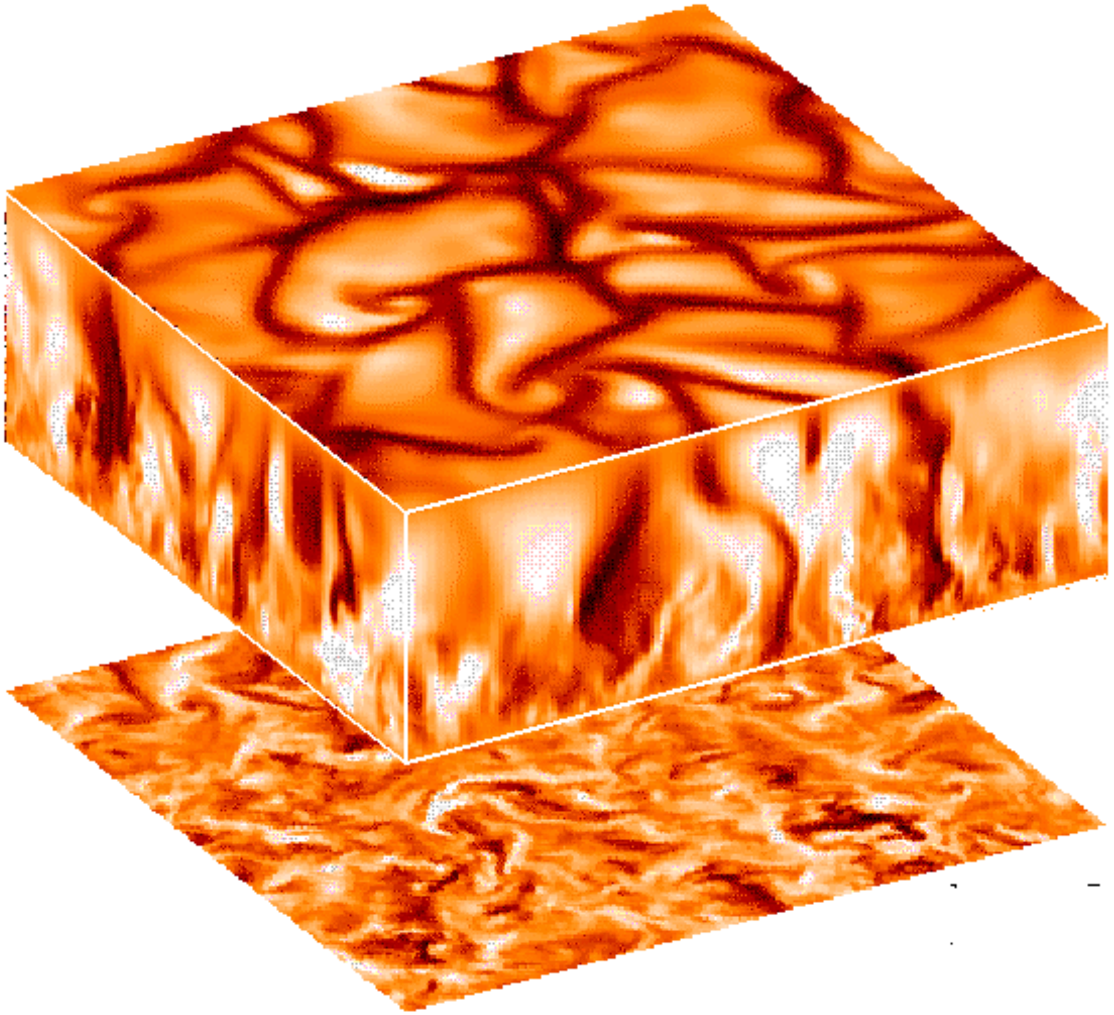
Very low mass Main sequence	Fully convective
Low mass Main sequence	Radiative core Convective surface
High mass Main sequence	Convective core Radiative surface
White dwarfs	Conductive

Convection


Core
Radiation from interior
200,000 km
30,000 km
Convection zone
1000 km
Photosphere

Convection

Convection is incredibly difficult to model accurately because fluid motions are very complicated. It is a 3D problem.



Convection

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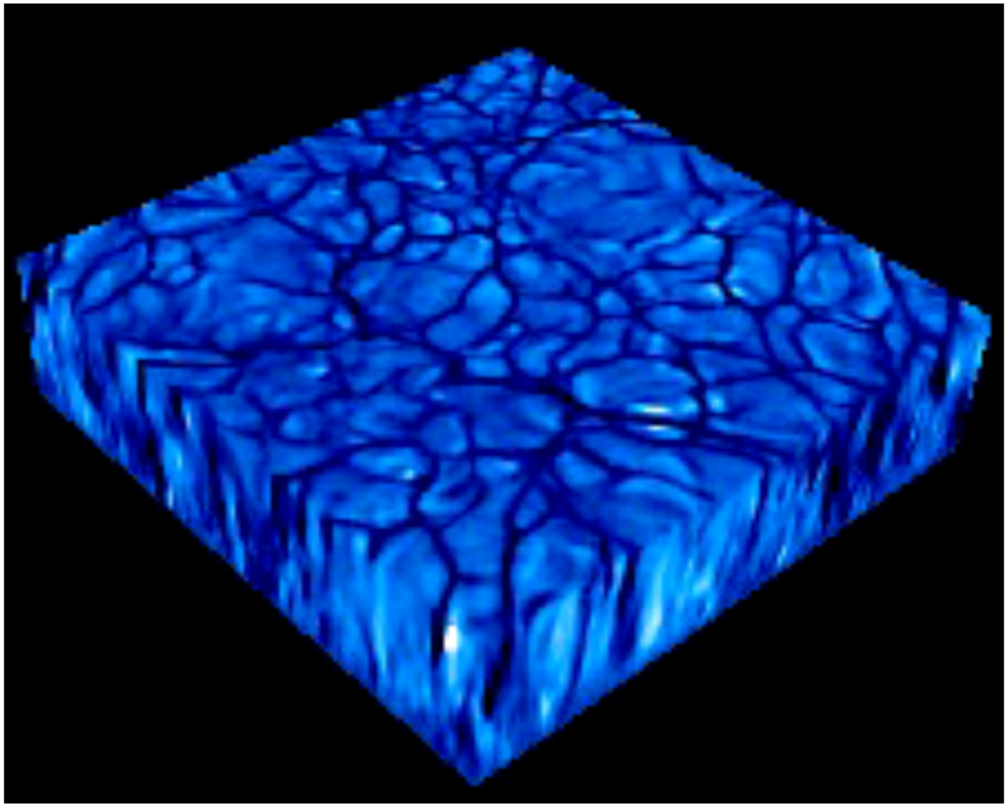
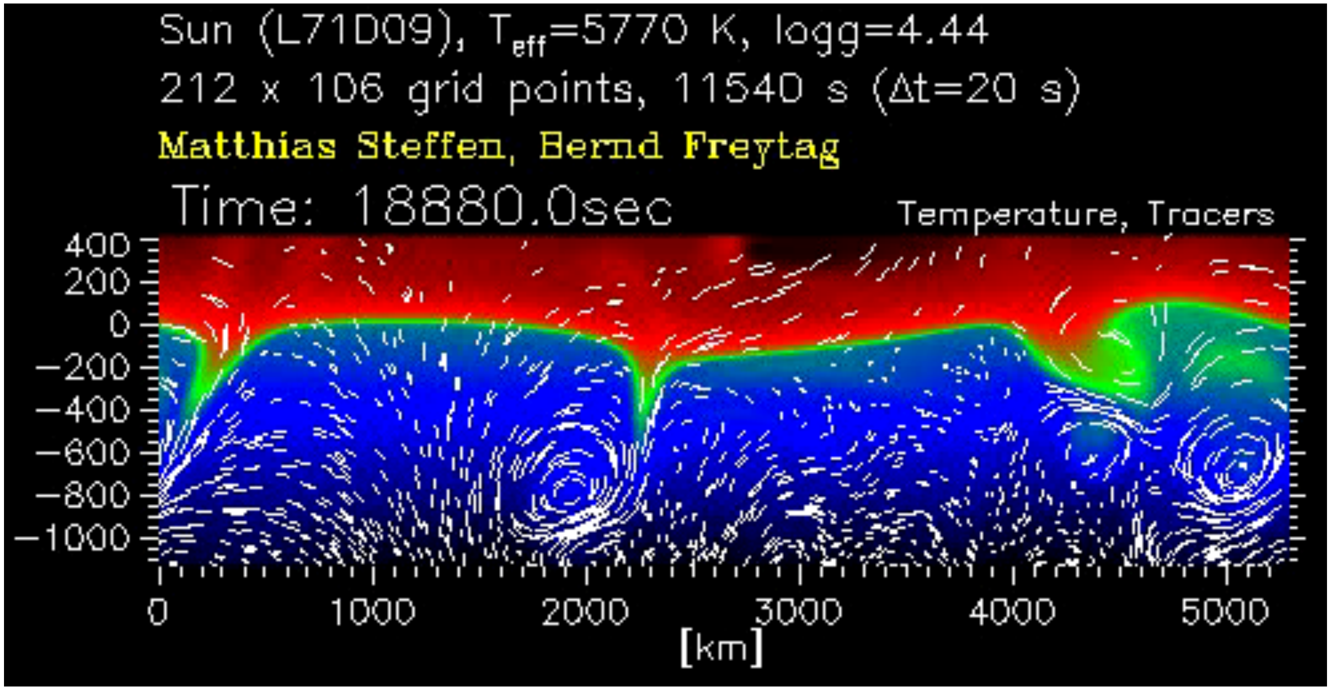


Convection

Convection is incredibly difficult to model accurately because fluid motions are very complicated. It is a 3D problem.



Convection: Energy Flux

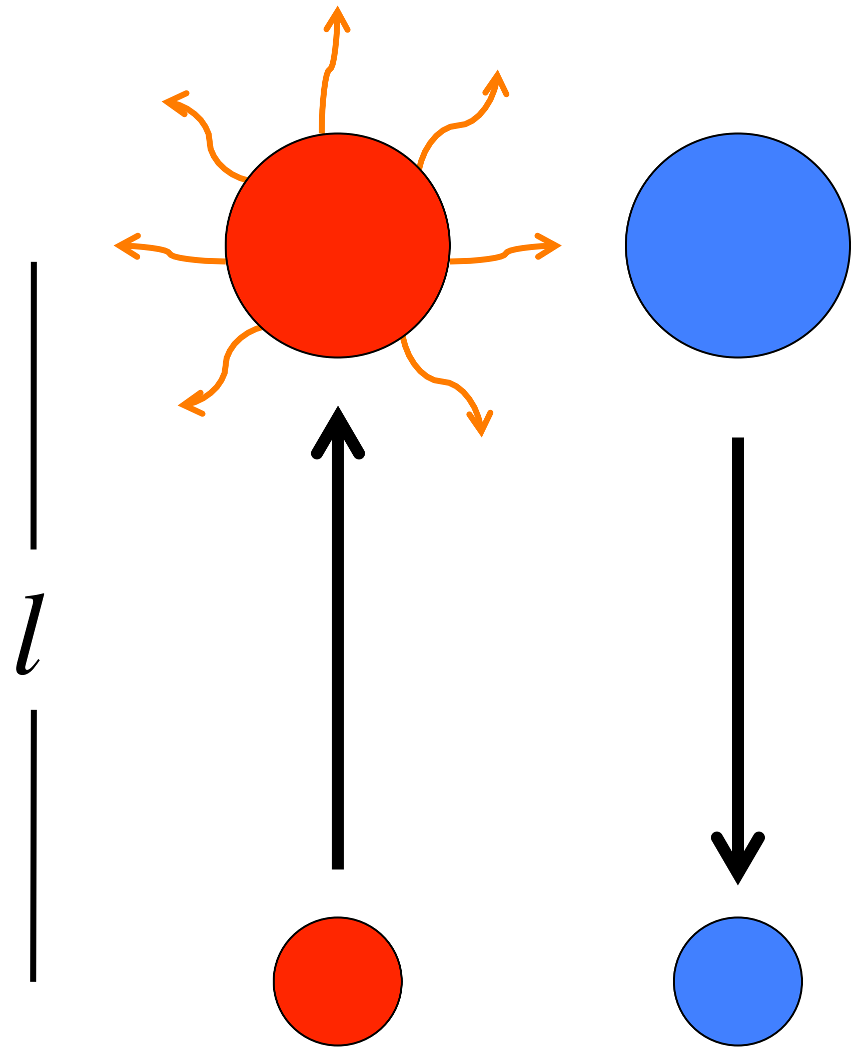
- No reliable analytic way to compute
- Lab experiments with fluids are not relevant
    - incompressible
    - too viscous
    - boundaries
- Hydrodynamic simulations are challenging
- Empirical simple model for spherically symmetric stellar models: Mixing Length Theory

Convection: Mixing Length Theory

1 parameter: length of mixing

Each mass element
rises or falls a distance
l adiabatically

After one mixing length,
The element thermalizes
with the local environment.



Convection: Mixing Length Theory

- l should scale with the pressure scale height

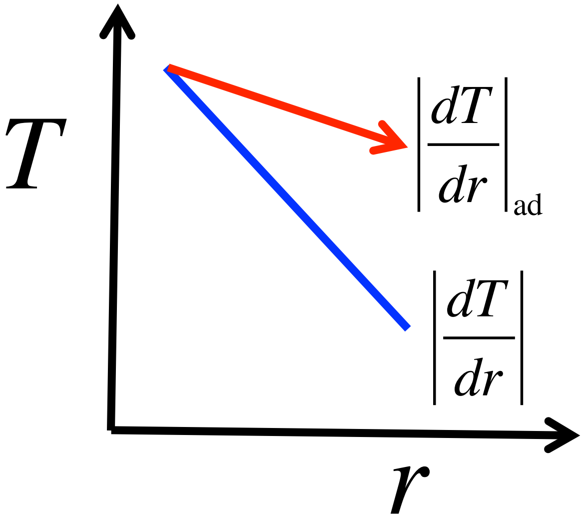
$$\lambda_P = P\left|\frac{dP}{dr}\right|^{-1}$$

(length over which pressure changes by 100%)

$$l = \alpha \cdot \lambda_P \qquad \alpha: \text{ free parameter } \quad (\alpha = 1.6 \pm 0.1)$$

- After rising adiabatically by a distance l, the mass element will be hotter than its surrounding gas by

$$\Delta T = \left(\left|\frac{dT}{dr}\right| - \left|\frac{dT}{dr}\right|_{\text{ad}}\right) \cdot l \equiv l \cdot \Delta\nabla T$$



Convection: Mixing Length Theory

- If the mass element thermalizes at constant pressure, the amount of heat per unit mass released is

$$\Delta Q = c_P \Delta T = c_P l \Delta \nabla T$$

- If the average velocity of the mass element is $\bar{v}$, the average excess heat flux is

$$F_{\text{conv}} = \rho \bar{v} \Delta Q = \rho \bar{v} c_P l \Delta \nabla T$$

$$\frac{\text{heat}}{\text{mass}} \times \frac{\text{mass}}{\text{volume}} \times \frac{\text{length}}{\text{time}} = \frac{\text{heat}}{\text{area} \cdot \text{time}}$$

- The mass element is accelerated due to buoyancy $\left(\rho < \rho_{\text{ambient}}\right)$

$$\Delta\rho = \left(\left| \frac{d\rho}{dr} \right| - \left| \frac{d\rho}{dr} \right|_{\text{ad}} \right) l \equiv l \Delta \nabla \rho \quad \rightarrow \overline{\Delta\rho} = \frac{1}{2} l \Delta \nabla \rho$$

Convection: Mixing Length Theory

- The average buoyant force per unit volume is

$$\bar{F} = g \cdot \overline{\Delta\rho} \quad = \frac{1}{2} g l \Delta\nabla\rho$$

where $g = \dfrac{Gm}{r^2}$

- The resulting acceleration is

$$\bar{a} = \frac{\bar{F}}{\rho} \quad = \frac{gl}{2\rho}\Delta\nabla\rho \quad = \frac{Gml}{2\rho r^2}\Delta\nabla\rho$$

- If the mass element starts at rest and has a constant acceleration

$$\left.\begin{aligned} v &= \bar{a} \cdot t \\ t &= \frac{l}{\bar{v}} = \frac{l}{\frac{1}{2}v} = \frac{2l}{v} \end{aligned}\right\} \rightarrow v = \left(2\bar{a}l\right)^{1/2} \quad \rightarrow \bar{v} = \frac{1}{2}\left(2\bar{a}l\right)^{1/2}$$

$$\rightarrow \bar{v} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\Delta\nabla\rho\right)^{1/2}$$

Convection: Mixing Length Theory

$$\boxed{F_{\text{conv}} = \rho \bar{v} c_P l \Delta\nabla T}$$

$$\boxed{\bar{v} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\Delta\nabla\rho\right)^{1/2}}$$

$$F_{\text{conv}} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\Delta\nabla\rho\right)^{1/2} \rho c_P l \Delta\nabla T$$

$$\Delta\nabla\rho \equiv \left(\left|\frac{d\rho}{dr}\right| - \left|\frac{d\rho}{dr}\right|_{\text{ad}}\right) \approx \frac{\rho}{T}\left(\left|\frac{dT}{dr}\right| - \left|\frac{dT}{dr}\right|_{\text{ad}}\right) \equiv \frac{\rho}{T}\Delta\nabla T$$

$$F_{\text{conv}} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\frac{\rho}{T}\Delta\nabla T\right)^{1/2} \rho c_P l \Delta\nabla T$$

$$\boxed{F_{\text{conv}} = \frac{l^2}{2} c_P \rho \left(\frac{Gm}{Tr^2}\right)^{1/2} \left(\Delta\nabla T\right)^{3/2}}$$

Convection: Mixing Length Theory

- How large an excess T gradient is needed to carry luminosity?

$$(\Delta\nabla T)^{3/2} = \frac{\dfrac{L_r}{4\pi r^2}}{\dfrac{l^2}{2} c_P \rho \left(\dfrac{Gm}{Tr^2}\right)^{1/2}} = \frac{L_r}{2\pi l^2 c_P \rho r (Gm/T)^{1/2}}$$

$$\text{e.g., at } \frac{m}{M} = 0.5,\ l = \lambda_P = 5 \times 10^9\,\text{cm},\ \text{solar values for } T, \rho, r, L_r, c_P$$

$$\Delta\nabla T \sim 10^{-10}\,\text{Kcm}^{-1} \qquad \left|\frac{dT}{dr}\right| \approx \frac{T_c}{R} \sim 10^{-4}\,\text{Kcm}^{-1}$$

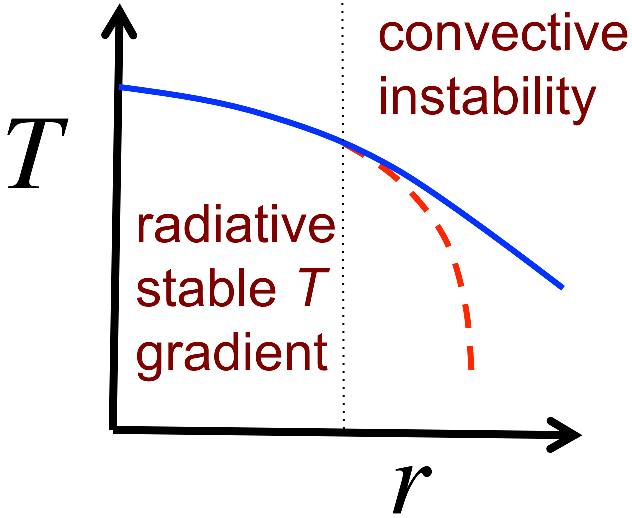
Requires an excess of only ~10^{-6} of the temperature gradient!

Convection: Mixing Length Theory

- When convection occurs in stellar interiors, the resulting temperature gradient equals the adiabatic gradient.

$$\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{\text{ad}} \cancel{- \Delta\nabla T}$$

$$\approx \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dT}$$



- If composition changes with radius, this may be different. e.g., if μ drops with r → heavier material is displaced into lighter material → greater stability.

Nuclear burning → stabilizing

Convection: Mixing Length Theory

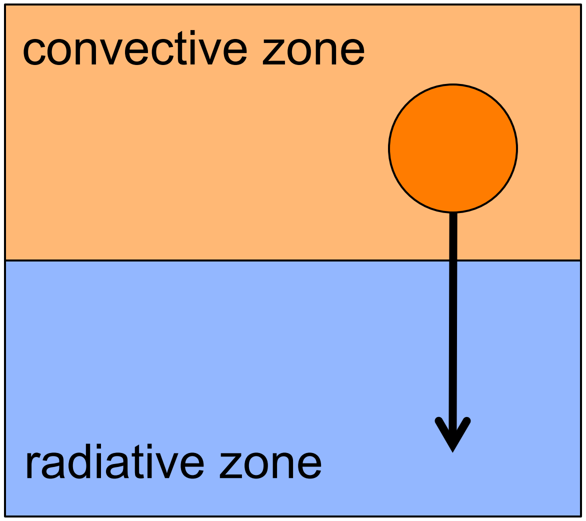
- When convection is efficient, the exact value of the mixing length l does not matter.

Near surface, λ_P is small → larger $\Delta\nabla T$ is needed.

- Stellar radius *R*, surface *T*, etc. depend on α

x2 change in α → few hundred K difference in T_{eff}.

Convection: Mixing Length Theory



Convective Overshooting

Do convective elements overshoot into stable regions due to inertia?

Naively: No because acceleration is small and braking is large in stable zone.

Simulations: Overshooting extends to significant fraction of λ_P

Alternative to Mixing Length model

"Full spectrum of turbulence"

Turbulent processes involve a range of length scales.

It is unclear if this works better than the one-parameter model.

Energy Generation: Gravitational Sources

In a static star, the energy source is thermonuclear

$$\frac{dL_m}{dm} = \varepsilon$$

In a contracting/expanding star, if the process is not adiabatic, there is an extra source/sink of energy so that

$$\frac{dL_m}{dm} \neq \varepsilon \qquad \text{Instead:} \quad \frac{dL_m}{dm} = \varepsilon + \varepsilon_{\text{grav}}$$

In real stars, contraction/expansion is a local process. e.g., core contracts while envelope expands → $\varepsilon_{\text{grav}}$ is a function of radius.

$$\varepsilon_{\text{grav}} = \frac{d\Omega}{dt \cdot dm} = \frac{d}{dt}\left(\frac{Gm}{r}\right) = -\frac{Gm}{r^2}\dot{r}$$

Energy Generation: Neutrino Losses

L_m excludes energy flux in neutrinos.

Matter at normal densities and temperatures is completely transparent to neutrinos → energy loss

neutrino cross-section: $\sigma_\nu \sim 10^{-44}\varepsilon_\nu^2 \text{ cm} \quad \left(\varepsilon_\nu : \text{ energy in MeV}\right)$

$$\lambda = \frac{1}{n\sigma_\nu} \qquad\qquad \lambda \sim 10^{20}\varepsilon_\nu^{-2}\rho^{-1}\text{cm}$$

It is only possible to get a short mean-free-path if the density is very high. This is only true in the cores of SN where neutrino pressure is important.

$$\boxed{\frac{dL_m}{dm} = \varepsilon + \varepsilon_{\text{grav}} - \varepsilon_\nu}$$

Nuclear Energy Generation

- Lighter nuclei with mass M_j fuse to form a heavier nucleus of mass M_y . The energy liberated is

$$E = \Delta M c^2 \quad = \left(\sum_j M_j - M_y \right) c^2$$

$$4 \times {}^1H \text{ (mass 1.0079 amu)} \rightarrow {}^4He \text{ (mass 4.0026 amu)}$$

$$\Delta M \sim 0.7\% \text{ or original masses, or 26.5MeV}$$

- The mass of a nucleus is not simply equal to the sum of its constituents (i.e., protons and neutrons). There is also binding energy that holds the constituents together.

Nuclear Energy Generation

- The binding energy of a nucleus is defined to be

$$E_B = \left(\text{mass of constituent nucleons } - \text{ mass of bound nucleus}\right)c^2$$

$$\boxed{E_B = \left[\left(A - Z\right)m_n + Zm_p - M_{\text{nuc}}\right]c^2}$$

m_n : neutron mass

m_p : proton mass

- The binding energy per nucleon is

$$f = \frac{E_B}{A}$$

- Binding energy = energy required to separate the nucleus to infinity against binding forces.

Nuclear Energy Generation

- To get energy from fusing lighter nuclei into heavier nuclei, the total binding energy of the light nuclei must be lower than that of the heavier nucleus. $\left(M_{\mathrm{nuc}}\text{for light} > M_{\mathrm{nuc}}\text{for heavy}\right)$

$$\text{e.g., } 3\times {}^{4}He \rightarrow {}^{12}C \qquad 3\times {}^{4}He: \quad \frac{3}{c^2}E_{B,He} = 3\times\left(2n+2p-M_{He}\right)$$

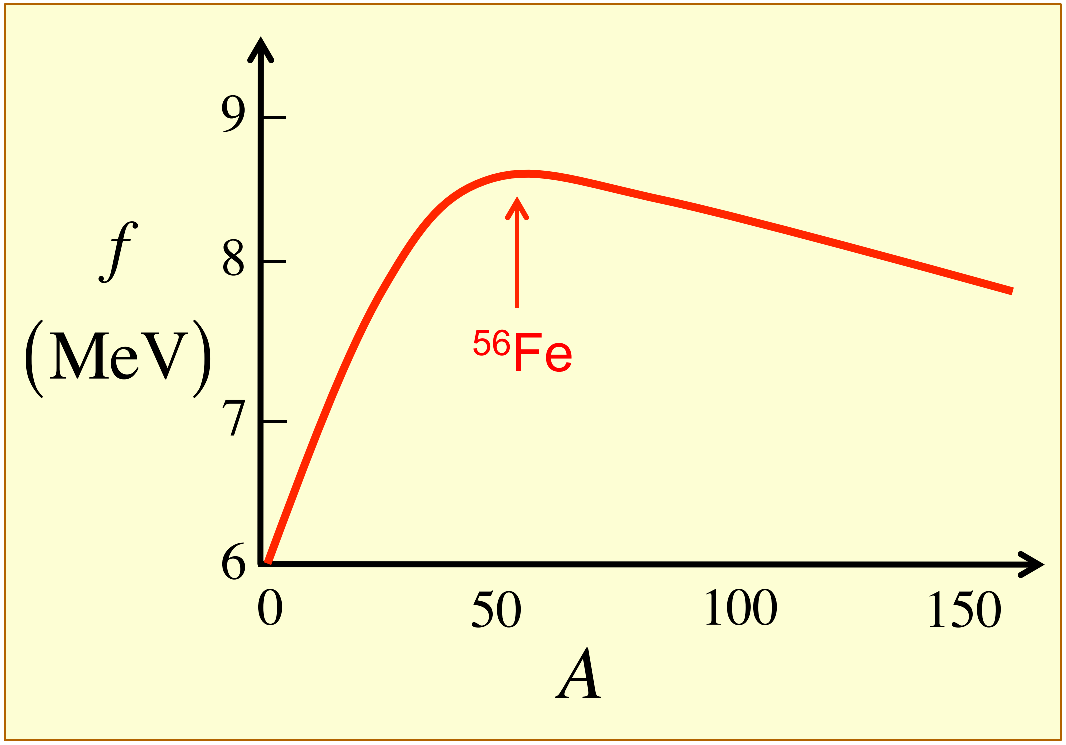
$$ {}^{12}C: \quad \frac{1}{c^2}E_{B,C} = \left(6n+6p-M_C\right)$$

$$3M_{He} > M_C \rightarrow \; 3E_{B,He} < E_{B,C} \quad \rightarrow \frac{E_{B,He}}{4} < \frac{E_{B,C}}{12}$$

- A nuclear reaction can release energy if $\frac{E_B}{A}$ for light element is less than that for heavy element.

Nuclear Energy Generation

- The most tightly bound nucleus is ^{56}Fe. Fusion of pure ^{1}H to ^{56}Fe yields ~8.5MeV per nucleon, the largest part of which (6.6MeV) is already obtained in fusion to ^{4}He.



- Fusion of elements with A<56 yields energy.

- Fission of elements with A>56 yields energy.

- Elements near ^{56}Fe are the most tightly bound and are thus not much use for energy production.

- If a star ends up with ^{56}Fe, energy production is over.

Nuclear Energy Generation: Coulomb Barrier

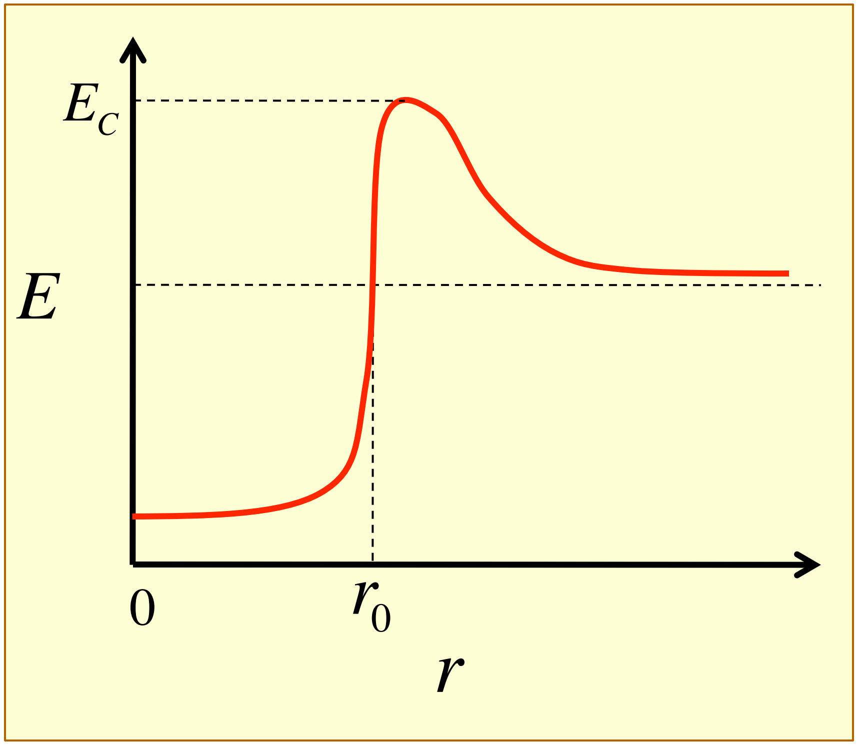
- To fuse, nuclei must surmount a Coulomb barrier.

- Nuclear forces dominate within a radius

$$r_0 = 1.44 \times 10^{-13} A^{1/3} \text{cm}$$

- For nuclei of charge Z_1 and Z_2 height of Coulomb barrier is

$$E_C = \frac{Z_1 Z_2 e^2}{r_0} \sim Z_1 Z_2 \text{ MeV}$$



- At T=10^7 K, the thermal energy kT is ~10^3eV
→ classically there are ZERO particles in a thermal distribution with sufficient energy to fuse at this T.

Nuclear Energy Generation: Tunneling

- Quantum tunneling is required for fusion to take place. Tunneling probability is

$$P = P_0 E^{-1/2} e^{-2\pi\eta} \qquad \eta = \left(\frac{m}{2}\right)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar E^{1/2}}$$

P_0 : depends on nuclei
m : reduced mass

P increases rapidly with energy
P decreases with Z_1, Z_2 → lightest elements can fuse at lowest temperatures.

- Higher energies and temperatures are needed to fuse heavier nuclei → well separated phases in which different elements burn during stellar evolution.

$$T \downarrow \quad \begin{array}{l} H \rightarrow He \\ He \rightarrow C,O \\ C,O \rightarrow Na,Ne,Si,P \end{array}$$

Nuclear Energy Generation: Resonances

- Most thermonuclear reactions in stars proceed through an intermediate state called the "compound nucleus"

$$\alpha + X \rightarrow Z^{*} \rightarrow Y + \beta$$

α : projectile (proton or α particle)

X : target nucleus

Z^{*} : compound nucleus in excited state

- Reactions happen in several steps:
  - Tunneling through the Coulomb barrier
  - Formation of a compound excited nucleus Z*, whose energy depends on the reaction and kinetic energy of reacting particles.
  - Decay of the Z* via emission of photons, neutrons, protons, alpha particles, electrons, etc.

Nuclear Energy Generation: Resonances

e.g., $$p + {}^{11}B \rightarrow {}^{12}C^{*} \rightarrow {}^{12}C^{**} + \gamma$$

$$\rightarrow {}^{11}B + p$$

$$\rightarrow {}^{11}C + n$$

$$\rightarrow {}^{12}N + e^{-} + \bar{\nu}_e$$

$$\rightarrow {}^{8}Be + {}^{4}He$$

- The probability of a given output depends on the decay lifetime over the sum of all possible lifetimes.

If τ_i is the lifetime of a particular output, then the probability of that output is:

$$P_i = \frac{1/\tau_i}{\sum_i 1/\tau_i} = \frac{\tau}{\tau_i} \quad \text{where} \quad \tau = \left(\sum_i 1/\tau_i\right)^{-1} : \text{total mean life of } {}^{12}\mathrm{C}^{*}$$

Nuclear Energy Generation: Resonances

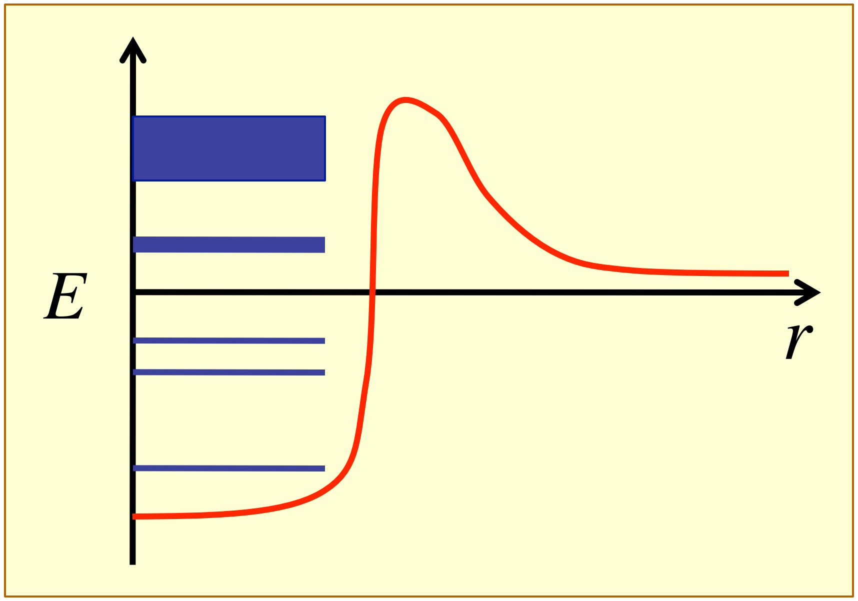
- Through the uncertainty principle, the energy width is $\Gamma_i \tau_i = \hbar$

$$P_i = \frac{\Gamma_i}{\sum_j \Gamma_j} = \frac{\Gamma_i}{\Gamma} \qquad \text{where } \Gamma : \text{total energy width of } {}^{12}\mathrm{C}^*$$

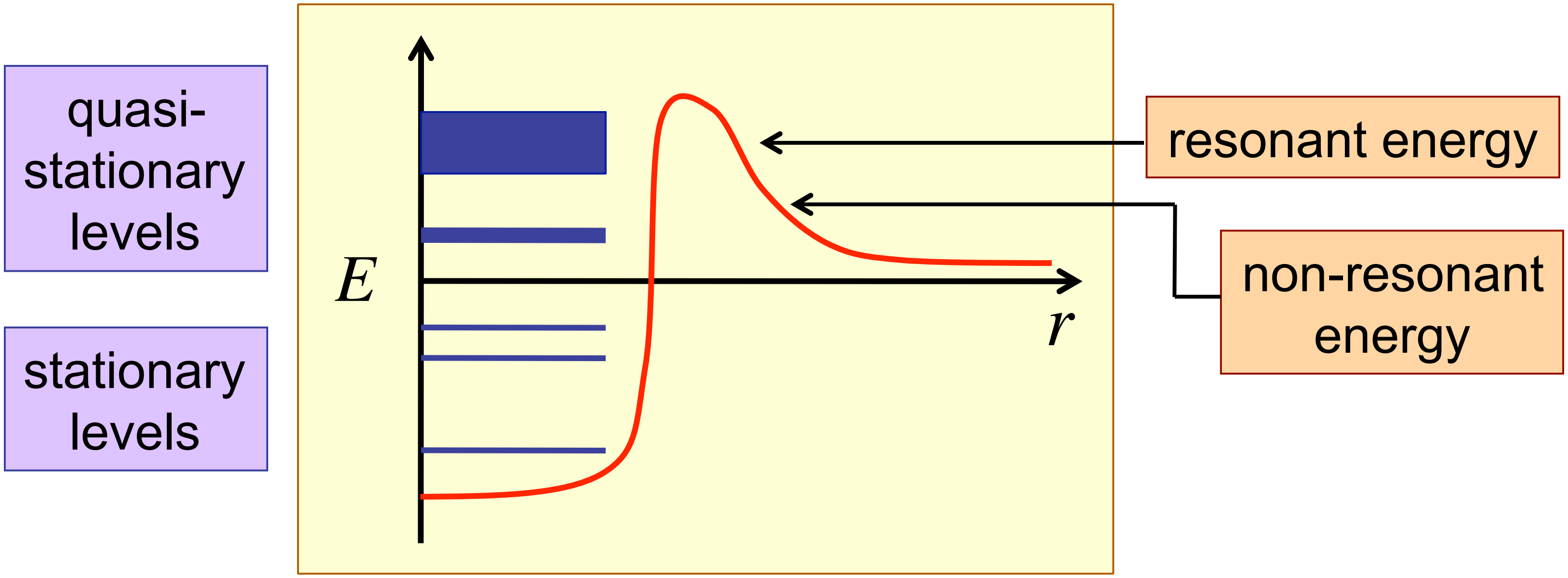
- The compound nucleus has

<u>Stationary</u> energy levels corresponding to excited atomic states that can decay via photon emission.

<u>Quasi-stationary</u> levels, which can decay via particles tunneling back through the Coulomb barrier.



Nuclear Energy Generation: Resonances



quasi-stationary levels

stationary levels

resonant energy

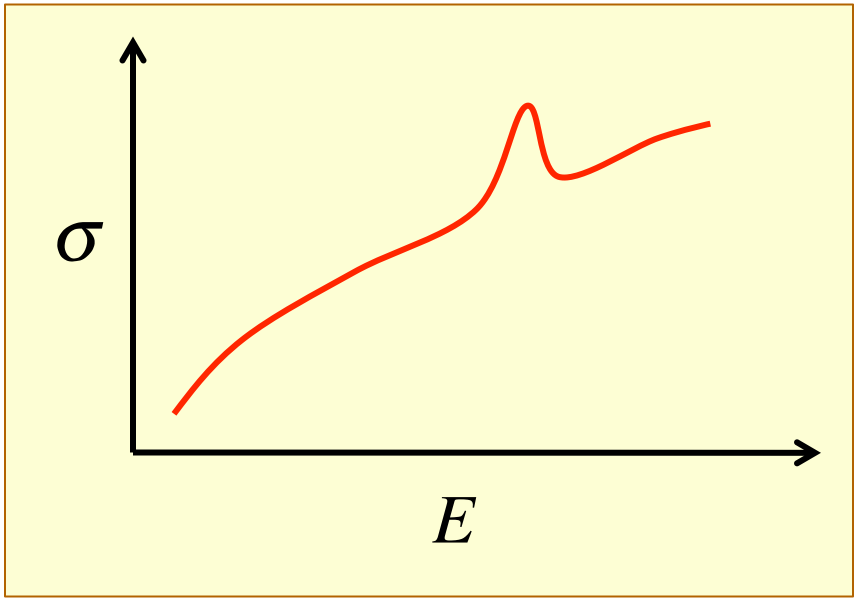
non-resonant energy

- The lifetimes of quasi-stationary states are shorter and they have broader energy ranges.

- At high energy, the width of states increases so that energy levels overlap → continuum of excited levels.

Nuclear Energy Generation: Resonances

- The cross-section for some astrophysically interesting reactions depends critically upon the energy level structure of the compound nucleus. At resonant energies, σ can be boosted by orders of magnitude.



- The maximum cross-section is the geometric cross-section

$$\pi\lambda^2 \propto E^{-1}$$

- Also, add dominant exponential tunneling factor

$$\boxed{\sigma(E) = S(E)E^{-1}e^{-2\pi\eta}}$$

S : astrophysical cross-section: varies very slowly with E

Nuclear Energy Generation: Reaction Rates

- The energy generation rate depends on the energy released per reaction times the reaction rate.

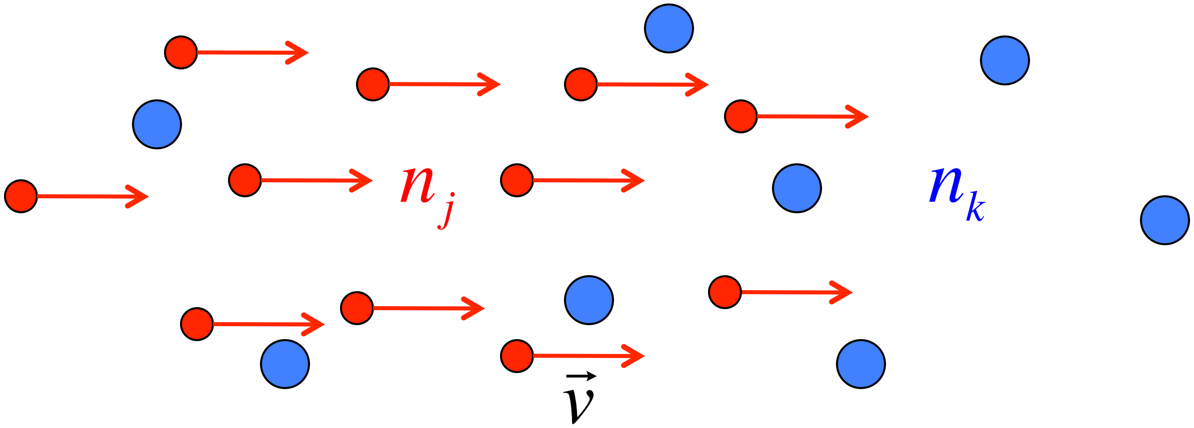
- Which energies of a particle contribute most to the total reaction rate?

  High *E* gives higher probability for fusion per reaction, but fewer potential reactions

- Why do nuclear reactions have a high *T* dependence?

Nuclear Energy Generation: Reaction Rates

- Consider particles of type *j* moving with velocity *v* relative to particles of type *k*. The number densities are n_j and n_k.



- The number of reactions per unit time per volume is:

$$\tilde{r}_{jk} = n_j \cdot n_k \cdot \sigma \cdot v$$

- To avoid double counting:

$$\boxed{\tilde{r}_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \sigma v}$$

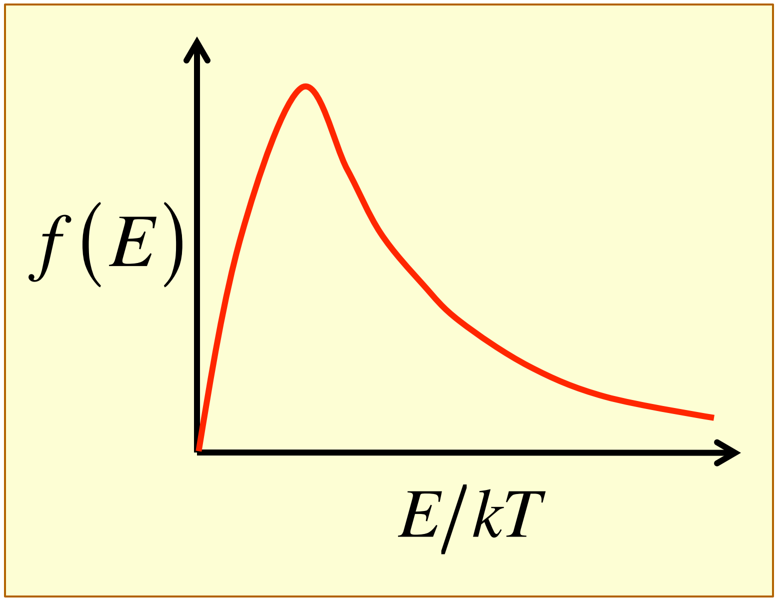
Nuclear Energy Generation: Reaction Rates

- Assume that both species have Maxwell-Boltzmann velocity distributions → the relative velocity is also Maxwellian.

$$E = \frac{1}{2} m v^2, \quad \text{where } m \text{ is the reduced mass:} \quad m = \frac{m_j m_k}{m_j + m_k}$$

- The fraction of all pairs with energy between *E* and *E*+*dE*:

$$\boxed{f(E)\,dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT}\,dE}$$



Nuclear Energy Generation: Reaction Rates

- The total reaction rate (per volume, per time) is the mono-energetic rate integrated over all energies and weighted by *f(E)*

$$r_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \langle \sigma v \rangle$$

where

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E) \cdot v \cdot f(E) dE$$

- Replace number densities with mass fractions: $X_i \rho = n_i m_i$

- If each reaction releases an amount of energy Q, then:

$$\boxed{\varepsilon_{jk} = \frac{1}{1+\delta_{jk}} \frac{Q}{m_j m_k} \rho X_j X_k \langle \sigma v \rangle}$$

$$\left(\frac{\text{energy}}{\text{time} \cdot \text{mass}} \right)$$

Nuclear Energy Generation: Reaction Rates

- All the temperature dependence is contained in $\langle \sigma v \rangle$

$$\sigma(E) = S(E)E^{-1}e^{-2\pi\eta}$$

$$v = \left(\frac{2E}{m}\right)^{1/2}$$

$$f(E)dE = \frac{2}{\sqrt{\pi}}\frac{E^{1/2}}{(kT)^{3/2}}e^{-E/kT}dE$$

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E)\cdot v\cdot f(E)dE$$

$$= \int_0^\infty S(E)\cancel{E}^{-1}e^{-2\pi\eta}\left(\frac{2}{m}\right)^{1/2}\cancel{E}^{1/2}\frac{2}{\sqrt{\pi}}\frac{\cancel{E}^{1/2}}{(kT)^{3/2}}e^{-E/kT}\,dE$$

$$= \frac{2^{3/2}}{(\pi m)^{1/2}}\frac{1}{(kT)^{3/2}}\int_0^\infty S(E)e^{-2\pi\eta}e^{-E/kT}\,dE$$

Nuclear Energy Generation: Reaction Rates

$$\langle \sigma v \rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) e^{-2\pi\eta} e^{-E/kT} \, dE$$

$$\bar{\eta} = 2\pi\eta E^{1/2} \quad = \pi (2m)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar}$$

$$\eta = \left(\frac{m}{2}\right)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar E^{1/2}}$$

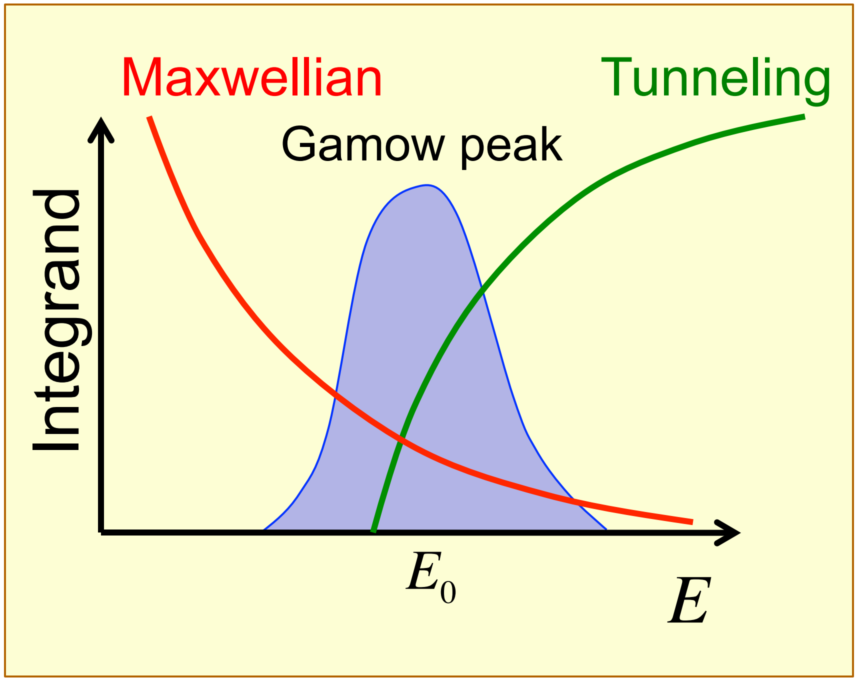
$$\langle \sigma v \rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)} \, dE$$

- Ignore *S(E)*. As *E* goes up, the energy distribution term (Maxwellian) drops, and the tunneling term increases.

Nuclear Energy Generation: Reaction Rates

$$\text{Integrand} = S(E)e^{\left(-E/kT-\bar{\eta}/E^{1/2}\right)}$$

- Integrand peaks at *E₀*, where *d*(Integrand)/*dE* = 0
- Since only a narrow range of *E* contributes to the integral, it is ok to assume $S(E) \approx S_0 = \text{const}$ (for non-resonant reactions)



$$\frac{d(\text{Integrand})}{dE} = e^{\left(-E/kT-\bar{\eta}/E^{1/2}\right)}\left(-\frac{1}{kT}+\frac{\bar{\eta}}{E}\frac{1}{2}E^{-1/2}\right)$$

- This is zero when: $\frac{\bar{\eta}}{E}\frac{1}{2}E^{-1/2} = \frac{1}{kT} \quad \rightarrow E_0 = \left(\frac{\bar{\eta}kT}{2}\right)^{2/3}$

Nuclear Energy Generation: Reaction Rates

- Evaluate the integral $$\int_0^\infty e^{\left(-E/kT-\bar{\eta}/E^{1/2}\right)}\,dE = \int_0^\infty e^{f(E)}\,dE$$

- Expand *f(E)* about the maximum at *E*=*E₀* using a series expansion truncated at the quadratic term

$$\bar{\eta} = \frac{2E_0^{3/2}}{kT}$$

$$f(E) = f(E_0) + \cancel{f'(E_0)(E-E_0)} + \frac{1}{2}f''(E_0)(E-E_0)^2 + \ldots$$

$$f(E_0) = -\frac{E_0}{kT} - \frac{\bar{\eta}}{E_0^{1/2}} \quad = -\frac{E_0}{kT} - \frac{2E_0}{kT} \quad = -3\frac{E_0}{kT}$$

$$f'(E_0) = -\frac{1}{kT} + \frac{\bar{\eta}}{2E_0^{3/2}} \quad = 0$$

$$f''(E_0) = -\frac{\bar{\eta}}{2E_0^3}\frac{3}{2}E_0^{1/2} \quad = -\frac{3\bar{\eta}}{4E_0^{5/2}} \quad = -\frac{3}{2E_0kT}$$

Nuclear Energy Generation: Reaction Rates

$$f(E) = f(E_0) + \frac{1}{2} f''(E_0)(E - E_0)^2 + \ldots$$

$$= -3\frac{E_0}{kT} + \frac{1}{2}\left(-\frac{3}{2E_0 kT}\right)(E - E_0)^2 + \ldots$$

$$= -3\frac{E_0}{kT} - \frac{3}{4E_0 kT} E_0^2 \left(\frac{E}{E_0} - 1\right)^2 + \ldots$$

$$= -3\frac{E_0}{kT} - \frac{3E_0}{4kT}\left(\frac{E}{E_0} - 1\right)^2 + \ldots$$

$$\tau = 3\frac{E_0}{kT}$$

$$= -\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2 + \ldots$$

$$\int_0^\infty e^{f(E)}\, dE = \int_0^\infty \exp\left[-\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2\right] dE$$

Nuclear Energy Generation: Reaction Rates

$$\int_0^\infty e^{f(E)}\,dE = \int_0^\infty \exp\left[-\tau - \frac{1}{4}\tau\left(\frac{E}{E_0}-1\right)^2\right]dE$$

$$= e^{-\tau}\int_0^\infty \exp\left[-\frac{\tau}{4}\left(\frac{E}{E_0}-1\right)^2\right]dE$$

$$x = \left(\frac{E}{E_0}-1\right)$$

$$dx = \frac{dE}{E_0}$$

$$= e^{-\tau}\int_0^\infty e^{-\frac{\tau}{4}x^2} E_0\,dx \qquad = E_0 e^{-\tau}\int_0^\infty e^{-\frac{\tau}{4}x^2}\,dx \qquad = E_0 e^{-\tau}\left(\frac{\pi}{\tau}\right)^{1/2}$$

$$= \frac{\tau kT}{3} e^{-\tau}\left(\frac{\pi}{\tau}\right)^{1/2} \qquad = \frac{\pi^{1/2}}{3} kT\tau^{1/2} e^{-\tau}$$

$$\boxed{\int_0^\infty e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)}\,dE = \frac{\pi^{1/2}}{3} kT\tau^{1/2} e^{-\tau}}$$

Nuclear Energy Generation: Reaction Rates

- Plug this into the expression for the cross-section

$$\langle \sigma v \rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{S_0}{(kT)^{3/2}} \int_0^{\infty} e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)} dE$$

$$\langle \sigma v \rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{1}{(kT)^{3/2}} \frac{\pi^{1/2}}{3} S_0 kT \tau^{1/2} e^{-\tau}$$

$$\boxed{\langle \sigma v \rangle = \frac{2^{2/3}}{3(m)^{1/2}} \frac{1}{(kT)^{1/2}} S_0 \tau^{1/2} e^{-\tau}}$$

$$\tau \sim \frac{E_0}{kT} \sim \frac{(kT)^{2/3}}{kT} = (kT)^{-1/3}$$

$$\boxed{\langle \sigma v \rangle \propto S_0 \tau^2 e^{-\tau}}$$

$$\rightarrow kT \sim \tau^{-3}$$

Nuclear Energy Generation: Reaction Rates

- Peak energy:

$$E_0 = \left(\frac{\bar{\eta} kT}{2}\right)^{2/3}$$

$$\bar{\eta} = \pi(2m)^{1/2}\frac{Z_1 Z_2 e^2}{\hbar}$$

$$\frac{E_0}{kT} = \left(\frac{\bar{\eta} kT}{2}\right)^{2/3}\frac{1}{kT} = \left(\frac{\pi(2m)^{1/2} Z_1 Z_2 e^2}{2\hbar}\right)^{2/3}(kT)^{-1/3}$$

$$m = \frac{m_j m_k}{m_j + m_k} = \frac{A_j A_k}{A_j + A_k} m_p$$

$$\frac{E_0}{kT} = \left(\frac{\pi^2 e^4 m_p}{2\hbar^2}\right)^{1/3}\left(\frac{A_j A_k}{A_j + A_k}\right)^{1/3}(Z_1 Z_2)^{2/3}(kT)^{-1/3}$$

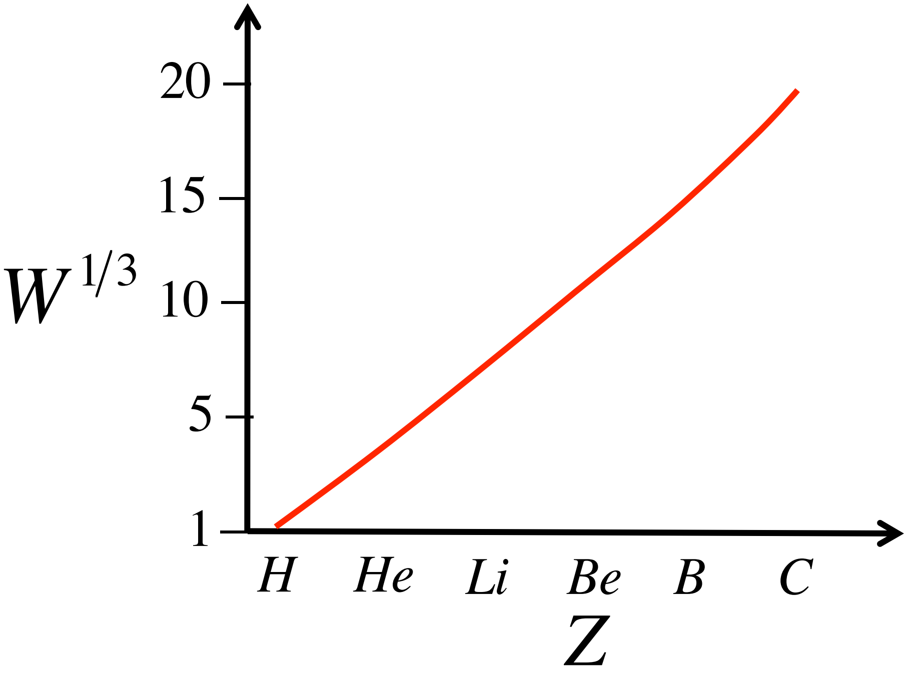
Nuclear Energy Generation: Reaction Rates

- Peak energy: $\dfrac{E_0}{kT} = \left(\dfrac{\pi^2 e^4 m_p}{2\hbar^2}\right)^{1/3} \left(\dfrac{A_j A_k}{A_j + A_k}\right)^{1/3} \left(Z_1 Z_2\right)^{2/3} \left(kT\right)^{-1/3}$

$$\boxed{\frac{E_0}{kT} \approx 6.6 W^{1/3} \left(\frac{T}{10^7\,\mathrm{K}}\right)^{-1/3}}$$

$$\boxed{W = Z_1^2 Z_2^2 \frac{A_j A_k}{A_j + A_k}}$$

The peak energy increases with W → separated phases of nuclear burning



Nuclear Energy Generation: Reaction Rates

$$\varepsilon_{jk} \sim \langle \sigma v \rangle \sim \tau^2 e^{-\tau}$$

$$\tau \sim T^{-1/3}$$

- Temperature dependence of energy generation rate

$$\boxed{\varepsilon_{jk} \sim T^{\nu}} \quad \rightarrow \nu = \frac{\partial \ln \varepsilon_{jk}}{\partial \ln T} \quad = \frac{\partial \ln \varepsilon_{jk}}{\partial \ln \tau} \frac{\partial \ln \tau}{\partial \ln T}$$

$$= \frac{\partial (2 \ln \tau - \tau)}{\partial \ln \tau} \frac{\partial (-\ln T/3)}{\partial \ln T} \quad = \left(2 - \frac{\partial \tau}{\partial \ln \tau} \right) \left(-\frac{1}{3} \right)$$

$$= (2 - \tau) \left(-\frac{1}{3} \right) \rightarrow \boxed{\nu = \frac{\tau}{3} - \frac{2}{3}} \boxed{= 6.6 W^{1/3} \left(\frac{T}{10^7 \mathrm{K}} \right)^{-1/3} - \frac{2}{3}}$$

$\nu \approx 5$ for lightest elements, rising to $\nu \approx 20$ for heavier nuclei.

Strong *T* dependence → stars must be stable to *T* fluctuations.

Nuclear Energy Generation: Reaction Rates

- We can use reaction rates to compute the time derivatives of mass fractions.

Number of reactions per volume, per time:

$$r_{jk} = \frac{1}{1+\delta_{jk}} X_j X_k \frac{\rho^2}{m_j m_k} \langle \sigma v \rangle$$

- Consider the reaction $A + B \rightarrow C$

The abundances of *A*, *B*, and *C* depend on $r_{A,B}$

- Each reaction yields one *C*, so the rate of change of the number density of *C* = $r_{A,B}$

$$\frac{dn_C}{dt} = r_{A,B} \quad \rightarrow \frac{dX_C}{dt}\frac{\rho}{m_C} = r_{A,B} \quad \rightarrow \frac{dX_C}{dt} = A_C \frac{m_p}{\rho} r_{A,B}$$

$$\rightarrow \frac{dX_C}{dt} = A_C \frac{m_p}{\rho} \frac{1}{1+\delta_{AB}} X_A X_B \frac{\rho^2}{m_A m_B} \langle \sigma v \rangle$$

Nuclear Energy Generation: Reaction Rates

$$\frac{dX_C}{dt} = A_C \frac{X_A X_B}{1+\delta_{AB}} \underbrace{\frac{\rho}{A_A A_B m_p} \langle \sigma v \rangle}_{r'_{A,B}} \quad = A_C \frac{X_A X_B}{1+\delta_{AB}} r'_{A,B}$$

e.g.,

$$\boxed{\begin{aligned} &{}^4He + {}^4He \rightarrow {}^8Be \\ &{}^8Be + {}^4He \rightarrow {}^{12}C \\ &{}^{12}C + {}^4He \rightarrow {}^{16}O \end{aligned}}$$

$$\frac{d\,{}^{16}O}{dt} = 16\left({}^4He\,{}^{12}C \cdot r'_{4,12}\right)$$

$$\frac{d\,{}^{12}C}{dt} = 12\left({}^4He\,{}^8Be \cdot r'_{4,8} - {}^4He\,{}^{12}C \cdot r'_{4,12}\right)$$

$$\frac{d\,{}^8Be}{dt} = 8\left(\frac{{}^4He\,{}^4He}{2} \cdot r'_{4,4} - {}^4He\,{}^8Be \cdot r'_{4,8}\right)$$

$$\frac{d\,{}^4He}{dt} = 4\left(-2\frac{{}^4He\,{}^4He}{2} \cdot r'_{4,4} - {}^4He\,{}^8Be \cdot r'_{4,8} - {}^4He\,{}^{12}C \cdot r'_{4,12}\right)$$

Nuclear Energy Generation: Reaction Rates

- In “equilibrium”, the mass fractions of ${}^{8}Be$ and ${}^{12}C$ are conserved.

$$\frac{d\,{}^{8}Be}{dt} = \frac{d\,{}^{12}C}{dt} = 0$$

$${}^{4}He + {}^{4}He \rightarrow {}^{8}Be$$
$${}^{8}Be + {}^{4}He \rightarrow {}^{12}C$$
$${}^{12}C + {}^{4}He \rightarrow {}^{16}O$$

$$\frac{{}^{4}He\,{}^{4}He}{2} \cdot r'_{4,4} = {}^{4}He\,{}^{8}Be \cdot r'_{4,8} \quad \rightarrow {}^{8}Be = \frac{1}{2}\,{}^{4}He\frac{r'_{4,4}}{r'_{4,8}}$$

$${}^{4}He\,{}^{8}Be \cdot r'_{4,8} = {}^{4}He\,{}^{12}C \cdot r'_{4,12} \quad \rightarrow {}^{12}C = {}^{8}Be\frac{r'_{4,8}}{r'_{4,12}} \quad = \frac{1}{2}\,{}^{4}He\frac{r'_{4,4}}{r'_{4,12}}$$

The proton-proton chain

- Nuclear reactions usually involve several steps.
- The rate is controlled by the rate of the slowest reaction in the chain.
- The total energy release is the sum of energies of the individual steps.

The first step in the p-p chain is: $^1H + {}^1H \rightarrow {}^2H + e^+ + \nu_e$

This reaction is non-resonant.

This reaction is slow because it involves a weak decay

While two protons are flashing past each other, one of them undergoes a weak decay into a neutron at that exact instant.

To make ^{4}He, we need 4 protons, two of which must be converted Into neutrons via either positron decays or e^- capture.

$$p \rightarrow n + e^+ + \nu_e \qquad\qquad p + e^- \rightarrow n + \nu_e$$

The proton-proton chain

The reaction rate is:

$$r_{pp} = 1.15 \times 10^9 T_9^{-2/3} X^2 \rho^2 \exp\left(-3.38 / T_9^{1/3}\right) cm^{-3} s^{-1}$$

The temperature sensitivity is:

$$\nu_{pp} = \frac{11.3}{T_6^{1/3}} - \frac{2}{3}$$

In sun: $T_6 \approx 15 \rightarrow \nu_{pp} \approx 4$

The lifetime of a proton against destruction is:

$$\tau_p = -\frac{n_p}{dn_p / dt} = \frac{n_p}{2r_{pp}} \approx 6 \times 10^9 yr$$ For center of Sun

The proton-proton chain

pp1

$$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_{e}$$

$$^{2}H + {}^{1}H \rightarrow {}^{3}He + \gamma$$

$$^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$$

pp2

First two reactions of pp1

$$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$$

$$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_{e}$$

$$^{7}Li + {}^{1}H \rightarrow {}^{4}He + {}^{4}He$$

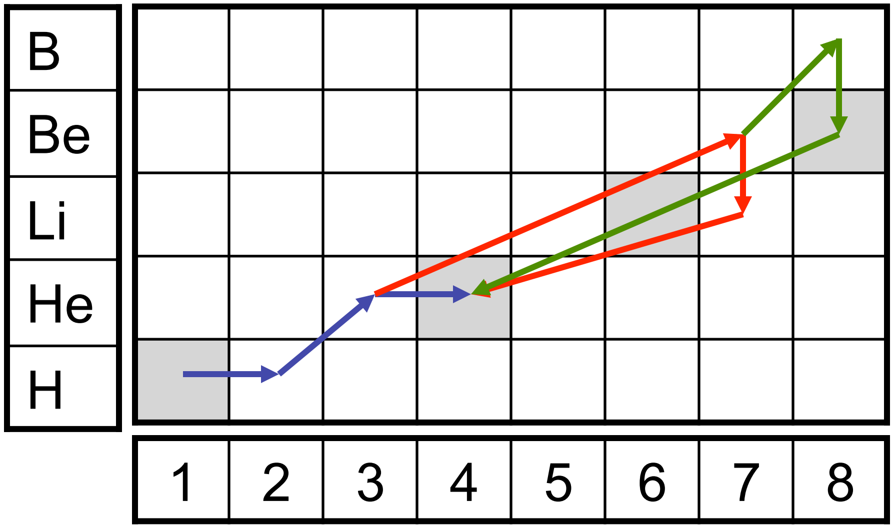
pp3

First three reactions of pp2

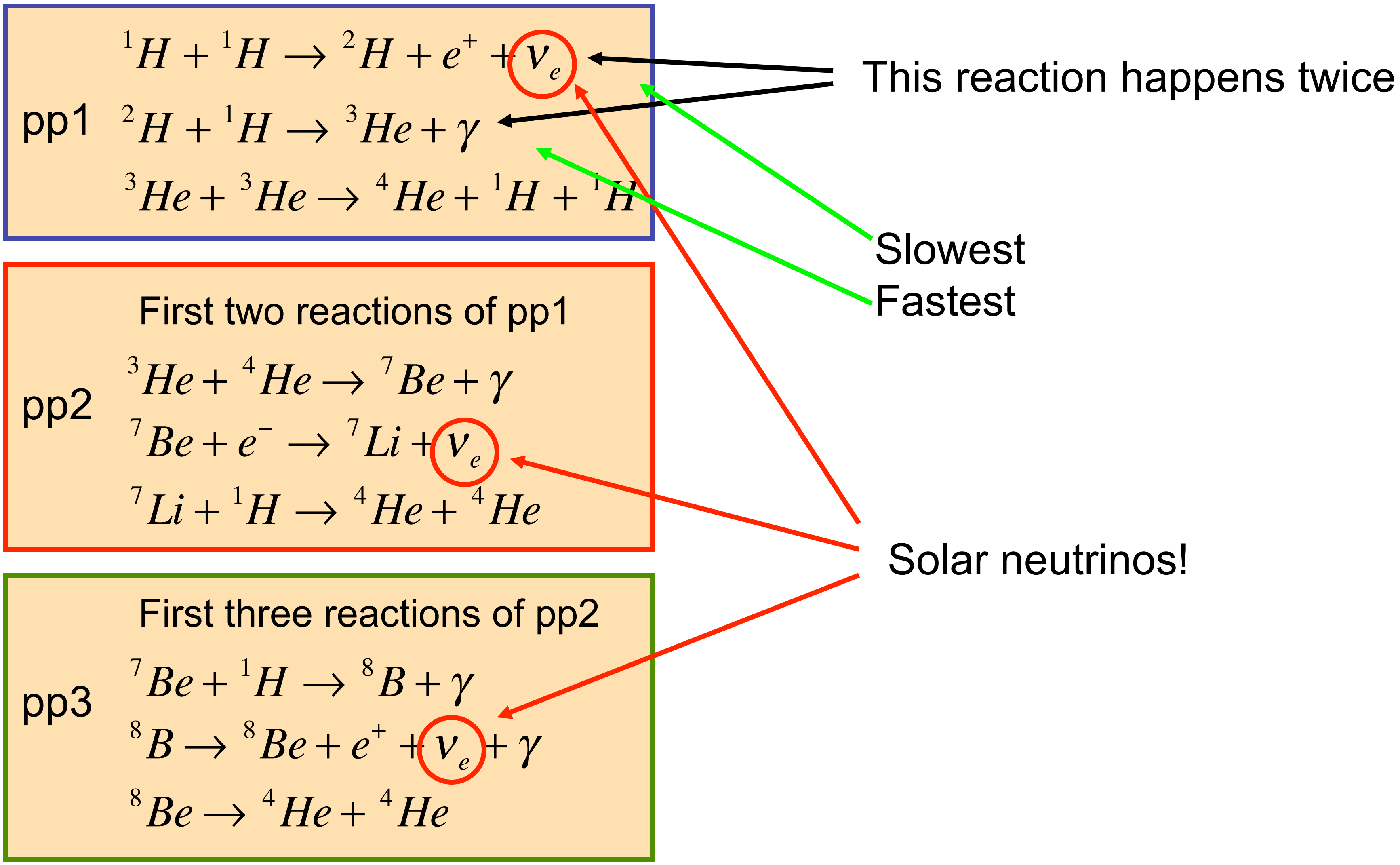
$$^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$$

$$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_{e} + \gamma$$

$$^{8}Be \rightarrow {}^{4}He + {}^{4}He$$



The proton-proton chain



pp1

$$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \enclose{circle}{\nu_e}$$

$$^{2}H + {}^{1}H \rightarrow {}^{3}He + \gamma$$

$$^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$$

pp2

First two reactions of pp1

$$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$$

$$^{7}Be + e^{-} \rightarrow {}^{7}Li + \enclose{circle}{\nu_e}$$

$$^{7}Li + {}^{1}H \rightarrow {}^{4}He + {}^{4}He$$

pp3

First three reactions of pp2

$$^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$$

$$^{8}B \rightarrow {}^{8}Be + e^{+} + \enclose{circle}{\nu_e} + \gamma$$

$$^{8}Be \rightarrow {}^{4}He + {}^{4}He$$

This reaction happens twice

Slowest

Fastest

Solar neutrinos!

The proton-proton chain

pp1

$$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_{e}$$

$$^{2}H + {}^{1}H \rightarrow {}^{3}He + \gamma$$

$$^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$$

pp2

First two reactions of pp1

$$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$$

$$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_{e}$$

$$^{7}Li + {}^{1}H \rightarrow {}^{4}He + {}^{4}He$$

pp3

First three reactions of pp2

$$^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$$

$$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_{e} + \gamma$$

$$^{8}Be \rightarrow {}^{4}He + {}^{4}He$$

pp1 is the most direct route but it involves the collision of two short lived ^{3}He nuclei.

Relative importance of chains thus depends on ^{3}He abundance.

As T increases, equilibrium abundance of ^{3}He decreases.

As T increases, importance of pp2 and pp3 relative to pp1 increases.

The proton-proton chain

pp1: $^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$ $\quad r_{3,3} \sim \left({}^{3}He\right)^{2} \langle \sigma v \rangle_{3,3}$

pp2: $^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$ $\quad r_{3,4} \sim {}^{3}He\,{}^{4}He \langle \sigma v \rangle_{3,4}$

pp2 will dominate when: $r_{3,4} > r_{3,3} \quad \rightarrow \quad \frac{{}^{4}He}{{}^{3}He} > \frac{\langle \sigma v \rangle_{3,3}}{\langle \sigma v \rangle_{3,4}} \gg 1$

pp2: $^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_{e}$ $\quad r_{7,-} \sim {}^{7}Be \times e^{-} \langle \sigma v \rangle_{7,-}$

pp3: $^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$ $\quad r_{7,+} \sim {}^{7}Be \times p^{+} \langle \sigma v \rangle_{7,+}$

pp3 will dominate when: $r_{7,+} > r_{7,-} \quad \rightarrow \quad \langle \sigma v \rangle_{7,+} > \langle \sigma v \rangle_{7,-}$

The proton-proton chain

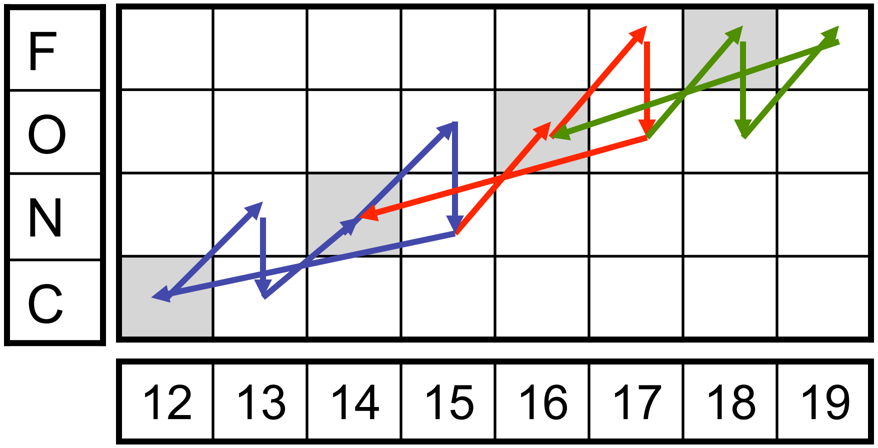
The energy generation rate is:

$$\varepsilon_{pp} = 2.4 \times 10^4 \frac{\rho X^2}{T_9^{2/3}} \exp\left(-3.38/T_9^{1/3}\right) ergs^{-1} g^{-1}$$

The CNO cycle

CNO1

$$^{12}C + {}^{1}H \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^{+} + \nu_{e}$$
$$^{13}C + {}^{1}H \rightarrow {}^{14}N + \gamma$$
$$^{14}N + {}^{1}H \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_{e}$$
$$^{15}N + {}^{1}H \rightarrow {}^{12}C + {}^{4}He$$



CNO2

$$^{15}N + {}^{1}H \rightarrow {}^{16}O + \gamma$$
$$^{16}O + {}^{1}H \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^{+} + \nu_{e}$$
$$^{17}O + {}^{1}H \rightarrow {}^{14}N + {}^{4}He$$

CNO3

$$^{17}O + {}^{1}H \rightarrow {}^{18}F + \gamma$$
$$^{18}F \rightarrow {}^{18}O + e^{+} + \nu_{e}$$
$$^{18}O + {}^{1}H \rightarrow {}^{19}F + \gamma$$
$$^{19}F + {}^{1}H \rightarrow {}^{16}O + {}^{4}He$$

The CNO cycle

CNO1

$$^{12}C + {}^{1}H \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^{+} + \nu_{e}$$
$$^{13}C + {}^{1}H \rightarrow {}^{14}N + \gamma$$
$$^{14}N + {}^{1}H \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_{e}$$
$$^{15}N + {}^{1}H \rightarrow {}^{12}C + {}^{4}He$$

This is the slowest reaction and sets the overall rate.

After long enough time, the most abundant nucleus will be ^{14}N
Most CNO → ^{14}N

Timescale to reach equilibrium is long

CNO2

$$^{15}N + {}^{1}H \rightarrow {}^{16}O + \gamma$$
$$^{16}O + {}^{1}H \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^{+} + \nu_{e}$$
$$^{17}O + {}^{1}H \rightarrow {}^{14}N + {}^{4}He$$

CNO3

$$^{17}O + {}^{1}H \rightarrow {}^{18}F + \gamma$$
$$^{18}F \rightarrow {}^{18}O + e^{+} + \nu_{e}$$
$$^{18}O + {}^{1}H \rightarrow {}^{19}F + \gamma$$
$$^{19}F + {}^{1}H \rightarrow {}^{16}O + {}^{4}He$$

The CNO cycle

The approximate energy generation rate is:

$$\varepsilon_{CNO} \approx 4.4 \times 10^{25} \frac{\rho XZ}{T_9^{2/3}} \exp\left(-15.228/T_9^{1/3}\right) ergs^{-1} g^{-1}$$

The temperature sensitivity is:

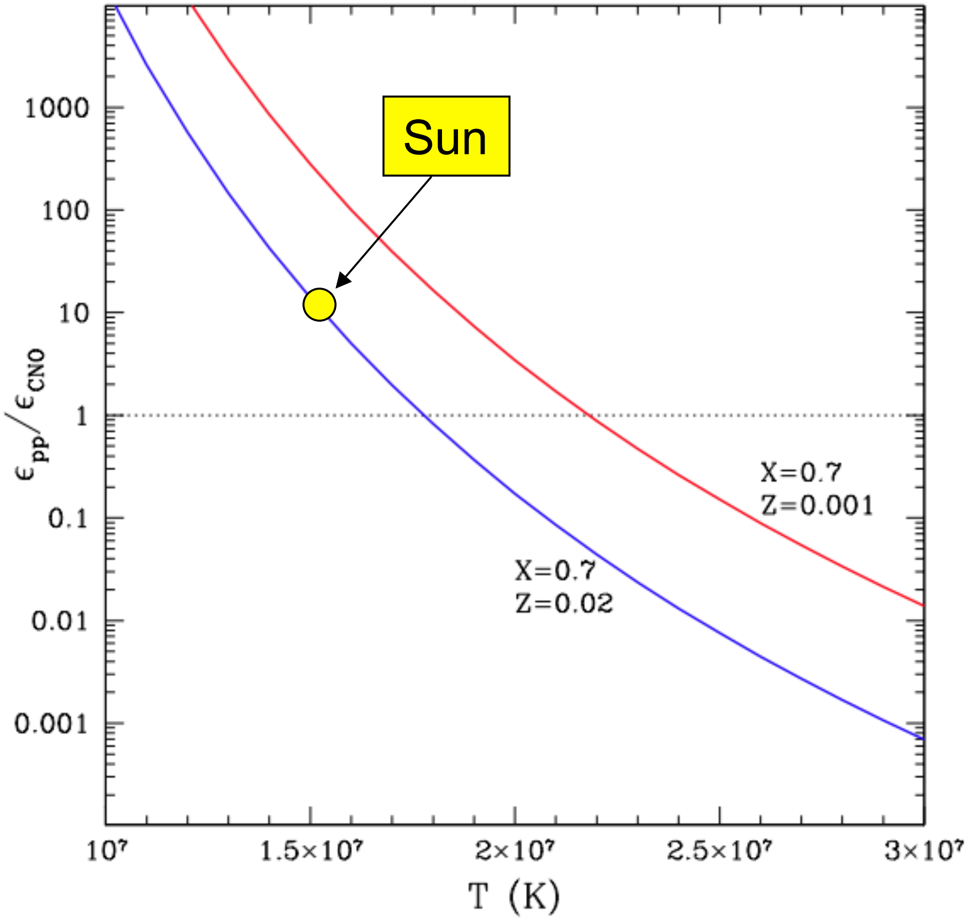
$$\nu_{CNO} = \frac{50.8}{T_6^{1/3}} - \frac{2}{3}$$

$$T_6 \approx 20 \rightarrow \nu_{CNO} \approx 18$$

pp chain vs. CNO cycle

CNO is favored over pp when:

- T is high
  Since heavier nuclei are involved
- Metallicity is high
  Since CNO nuclei are needed



He Burning (triple α)

$$^4He + ^4He \rightarrow ^8Be$$

This is the inverse of the final reaction in the pp3 chain.

The reaction is endothermic, absorbing 92 keV of energy. Requires about this much energy in Gamow peak

$$\frac{E_0}{kT} \approx 6.6 W^{1/3} T_7^{-1/3} \qquad W = Z_j^2 Z_k^2 \frac{A_j A_k}{A_j + A_k} \qquad E_0 = 92 keV \rightarrow T = 1.15 \times 10^8 K$$

Lower limit for He burning

- When T>10^{8}K, ^{8}Be is created roughly as fast as it is destroyed (via inverse reaction)

We can find equilibrium abundance via $\langle \sigma v \rangle$, but we can also use a Saha equation for nuclei:

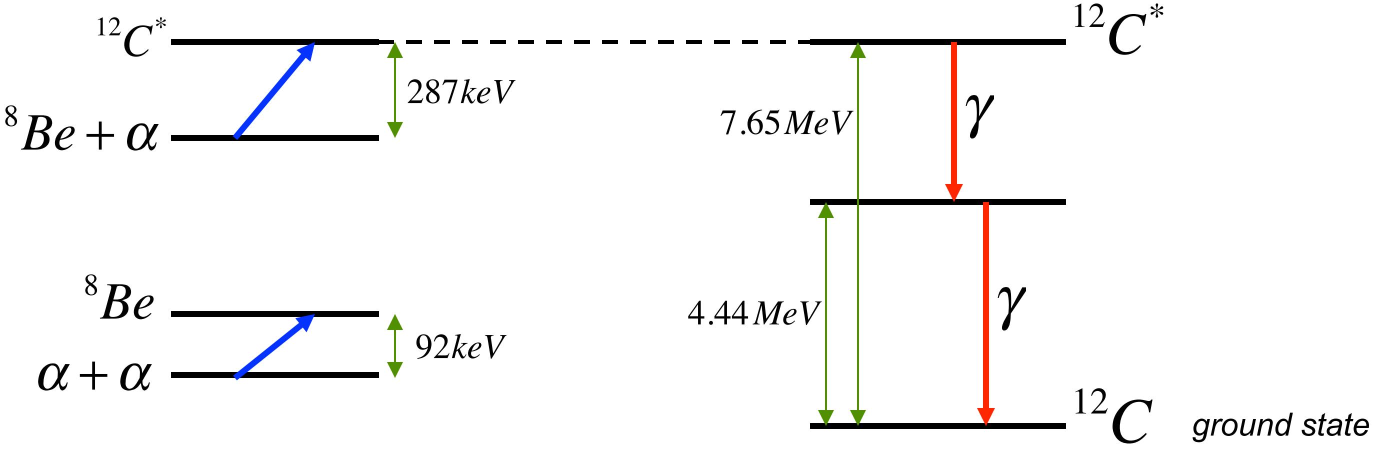
$$\frac{n_\alpha^2}{n_{^8Be}} = \left(\frac{\pi m_\alpha kT}{h^2} \right)^{3/2} e^{-Q/kT} \qquad Q = 92 keV \qquad T = 10^8 K \rightarrow \frac{n_{Be}}{n_\alpha} = 7 \times 10^{-9}$$

He Burning

$$^{8}Be + {}^{4}He \rightarrow {}^{12}C + \gamma + \gamma$$

The reaction is exothermic and resonant, proceeding via an excited state $^{12}C^{*}$

Most $^{12}C^{*}$ decay straight back to ^{8}Be + ^{4}He, but some emit a photon to form a stable ^{12}C



He Burning

The approximate energy generation rate is:

$$\varepsilon_{3\alpha} = 5 \times 10^{8} \frac{\rho^{2} Y^{3}}{T_9^{3}} \exp(-4.4/T_9) ergs^{-1} g^{-1}$$

The temperature sensitivity is:

$$\nu_{3\alpha} = \frac{4.4}{T_9} - 3$$

$$T_9 = 0.1 \rightarrow \nu_{3\alpha} \approx 40$$

He Burning

The effect of He burning on the structure of the star depends on the central conditions at the moment of ignition.

- Low mass stars ($M < 0.4M_{sun}$) develop a fully degenerate Helium core before the temperature rises to the He ignition threshold. The core cannot contract or heat up further → no He burning → He white dwarf

- High mass stars ($M > 1.5M_{sun}$) ignite He while the density is well below the degeneracy threshold ($\rho << 10^6$g cm^{-3}) → No large structural changes

- Intermediate mass stars ignite He under conditions of partial degeneracy. This allows a runaway reaction: He flash

He Flash

- Under degenerate conditions, pressure is a function of ρ only, not T.
- Upon He ignition, the energy release leads to a T rise. However, this does not lead to a P rise and hence to expansion and consequent reduction in ρ and T, as in a non-degenrate gas.
- Higher T increases the rate of nuclear reactions further → runaway

Non-degenerate core: He ignition → T ↑ → P ↑ → expand → T ↓ stable

Degenerate core: He ignition → T ↑ → ε ↑ → T ↑ unstable

- This continues until T is high enough in the core to lift degeneracy. The core then expands rapidly until a new stable He burning structure is attained.

Heavier elements

- Once enough ^{12}C is formed by the triple-α reaction, further captures of α particles occur simultaneously with the Carbon forming reaction.

$$^{12}C+{}^{4}He\rightarrow{}^{16}O+\gamma$$

$$^{16}O+{}^{4}He\rightarrow{}^{20}Ne+\gamma$$

(Reactions beyond ^{20}Ne are rare.)

- He burning ---> C, O, Ne. If T>10^9K, further reactions can occur. Since binding energies of heavy nuclei are comparable, a wide range of reactions are possible.

$$^{12}C+{}^{12}C\rightarrow{}^{20}Ne+{}^{4}He$$

$$^{12}C+{}^{12}C\rightarrow{}^{23}Na+p$$

Most important C burning

$$^{16}O+{}^{16}O\rightarrow{}^{31}P+p$$

Most common O burning

$$^{16}O+{}^{16}O\rightarrow{}^{28}Si+{}^{4}He$$

$$^{16}O+{}^{16}O\rightarrow{}^{31}S+n$$

- Si burning starts at T~3x10^9K
- over long timescales → ^{56}Fe
- over short timescales → ^{56}Ni (*e.g., in SN*)

Summary of Results

1. Mass Conservation

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

eulerian

$$\boxed{\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}}$$

lagrangian

Summary of Results

2. Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2}$$

eulerian

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

lagrangian

Need equation of state: $P = f\left(\rho, T, X_i\right)$

Summary of Results

3. Energy Generation

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$$

eulerian

$$\boxed{\frac{dL_m}{dm} = \varepsilon}$$

lagrangian

Need nuclear physics: $\varepsilon = f\left(\rho, T, X_i\right)$

Summary of Results

4. Energy Flow: Radiation

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acr^2 T^3} \qquad \frac{dT}{dm} = -\frac{3\kappa L_m}{64\pi^2 acr^4 T^3}$$

eulerian *lagrangian*

Need opacity: $\kappa = f\left(\rho, T, X_i\right)$

Summary of Results

4. Energy Flow: Convection

$$\frac{dT}{dr} = -\left(1 - \frac{1}{\Gamma_2}\right)\frac{Gm\rho T}{r^2 P}$$

eulerian

$$\frac{dT}{dm} = -\left(1 - \frac{1}{\Gamma_2}\right)\frac{GmT}{4\pi r^4 P}$$

lagrangian

Need adiabatic exponent: $\Gamma_2 = f\left(\rho, T, X_i\right)$

Summary of Results

Equation of State (non-degenerate matter)

$$\boxed{P = \frac{1}{3}aT^4 + \frac{N_A k}{\mu}\rho T}$$

Mean molecular weight: $\dfrac{1}{\mu} = \dfrac{1}{\mu_i} + \dfrac{1}{\mu_e} = \sum_i \dfrac{X_i}{A_i} + \sum_i \dfrac{Z_i X_i y_i}{A_i}$

Summary of Results

Ionization fractions y_i

Saha equation :

$$\frac{n^+ n_e}{n^0} = \frac{2G^+}{G^0} \frac{\left(2\pi m_e kT\right)^{3/2}}{h^3} \exp\left(-\frac{\chi}{kT}\right)$$

e.g., Hydrogen only :

$$\frac{y^2}{1-y} = \frac{1}{N_A \rho} \frac{\left(2\pi m_e kT\right)^{3/2}}{h^3} \exp\left(-\frac{\chi_H}{kT}\right)$$

Summary of Results

Thermodynamics

Adiabatic index :

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad}$$

$$\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2} \qquad \beta = \frac{P_g}{P_{tot}}$$

(assuming neutral or fully ionized gas)

Summary of Results

Opacity: approximations

$$\frac{1}{\kappa} \approx \frac{1}{\kappa_{H^-}} + \frac{1}{\kappa_e + \kappa_{ff} + \kappa_{bf}}$$

$$\kappa_e = 0.2\left(1 + X\right) cm^2 g^{-1}$$

(assuming fully ionized and no metals)

$$\kappa_{ff} \approx 4 \times 10^{22} \left(X + Y\right)\left(1 + X\right) \rho T^{-3.5} cm^2 g^{-1}$$

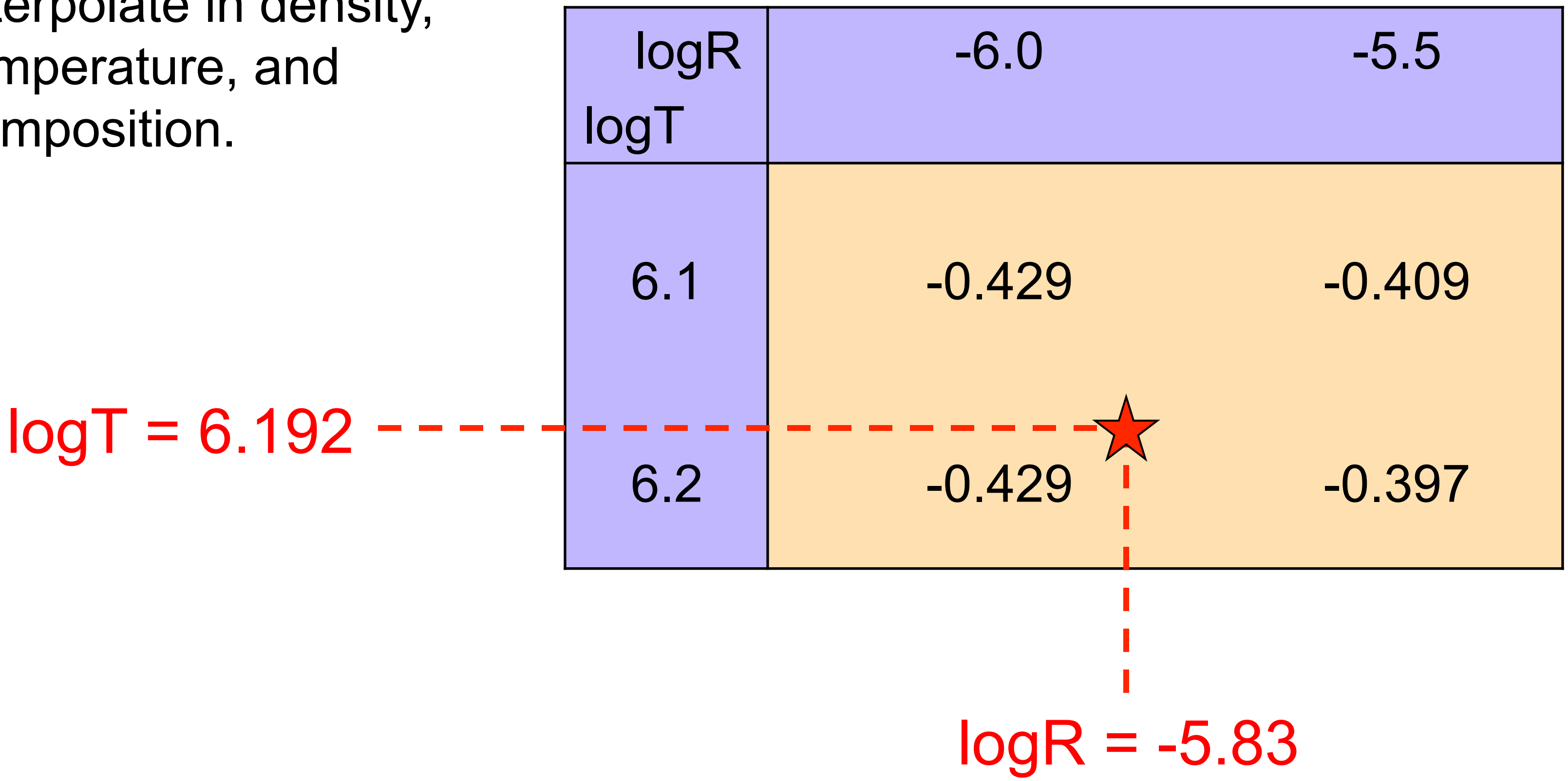
$$\kappa_{bf} \approx 4 \times 10^{25} Z \left(1 + X\right) \rho T^{-3.5} cm^2 g^{-1}$$

$$\kappa_{H^-} \approx 2.5 \times 10^{-31} \left(Z/0.02\right) \rho^{1/2} T^9 cm^2 g^{-1}$$

Summary of Results

Opacity: tables

Interpolate in density, temperature, and composition.



logR logT	-6.0	-5.5
6.1	-0.429	-0.409
6.2	-0.429	-0.397

logT = 6.192

logR = -5.83

Summary of Results

Convection

Convection happens when: $\left|\frac{dT}{dr}\right|_{rad} > \left|\frac{dT}{dr}\right|_{ad}$

$$L_m > \frac{16\pi acG}{3\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

Summary of Results

Energy generation

$$\varepsilon = \varepsilon_{pp} + \varepsilon_{CNO} + \varepsilon_{3\alpha}$$

$$\varepsilon_{pp} = 2.4 \times 10^{4} \frac{\rho X^{2}}{T_9^{2/3}} \exp\left(-3.38/T_9^{1/3}\right) ergs^{-1} g^{-1}$$

$$\varepsilon_{CNO} \approx 4.4 \times 10^{25} \frac{\rho XZ}{T_9^{2/3}} \exp\left(-15.228/T_9^{1/3}\right) ergs^{-1} g^{-1}$$

$$\varepsilon_{3\alpha} = 5 \times 10^{8} \frac{\rho^{2} Y^{3}}{T_9^{3}} \exp\left(-4.4/T_9\right) ergs^{-1} g^{-1}$$

Stellar Models

Given M and X,Y,Z, solve structure equations

$$\frac{dr}{dm} \qquad \frac{dP}{dm} \qquad \frac{dT}{dm} \qquad \frac{dL_m}{dm}$$

to get r, P, T, L_m as a function of m

Need to know:

$$\rho(P,T,X,Y,Z) \qquad \varepsilon(\rho,T,X,Y,Z)$$

$$\kappa_R(\rho,T,X,Y,Z) \qquad \Gamma_2(\rho,T,X,Y,Z)$$

Stellar Models

CENTER	SURFACE
$m = 0$	$m = M$
$r = 0$ ✓	$r = R$
$P = P_c$	$P \approx 0$ ✓
$T = T_c$	$T \approx 0$ ✓
$L_m = 0$ ✓	$L_m = L$

Stellar Models

$$\frac{dr}{dm} = \frac{1}{4\pi \boxed{r^2} \rho}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi \boxed{r^4}}$$

$$\frac{dL_m}{dm} = \varepsilon$$

$$\frac{dT}{dm} = -\frac{3\kappa L_m}{64\pi^2 a c \boxed{r^4} T^3}$$

$$\frac{dT}{dm} = -\left(1 - \frac{1}{\Gamma_2}\right)\frac{GmT}{4\pi \boxed{r^4} P}$$

- Some of these equations are indeterminant at center.
- It is better to start the integration at a very small m>0.

Stellar Models

Central boundary conditions

At a very small m>0:

$$r \neq 0 \qquad m = \frac{4}{3}\pi r^3 \rho_c \to \boxed{r = \left(\frac{3m}{4\pi\rho_c}\right)^{1/3}}$$

$$L_m \neq 0 \qquad \boxed{L_m = \varepsilon_c m}$$

Stellar Models

Central boundary conditions

- We can get boundary conditions for P and T by expanding and demanding that their derivatives are zero at r=0

$$P(r) = P(0) + \cancelto{0}{P'}(0)(r-0) + \frac{1}{2}P''(0)(r-0)^2$$

$$= P_c + P''(0)\frac{r^2}{2}$$

$$P'(r) = -\frac{Gm\rho}{r^2}$$

$$\boxed{P(r) = P_c - \frac{2}{3}\pi G\rho_c^2 r^2}$$

Stellar Models

Central boundary conditions

- Similarly,

$$T(r) = T(0) + \cancelto{0}{T'(0)}(r-0) + \frac{1}{2}T''(0)(r-0)^2$$

$$= T_c + T''(0)\frac{r^2}{2}$$

$$T'(r) = -\frac{3\kappa_R \rho L_r}{16\pi acr^2T^3}$$

$$T'(r) = \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

Radiative

$$T(r) = T_c - \frac{\kappa_c \rho_c^2 \varepsilon_c}{8acT_c^3}r^2$$

Convective

$$T(r) = T_c - \left(1 - \frac{1}{\Gamma_{2,c}}\right)\frac{2\pi G\rho_c^2 T_c}{P_c}r^2$$

Stellar Models

Surface boundary conditions

- We can do better than T=P=0 at surface.

guess R and L:

$$L = 4\pi R^2 \sigma T_s^4 \rightarrow \quad \boxed{T_s = \left(\frac{L}{4\pi R^2 \sigma} \right)^{1/4}}$$

Guess ρ very small, i.e. 10^{-5} g cm^{-3}:

$$\boxed{P_s = \frac{1}{3} a T_s^4 + \frac{N_A k}{\mu} \rho_s T_s}$$

Stellar Models

Steps in constructing a stellar model

1. Compute $\rho,\ \kappa,\ \varepsilon,\ \Gamma_2$ as a function of $P,\ T,\ X,\ Y,\ Z$

   DENSITY

   - Beware of P dropping below $\frac{1}{3}aT^4$

   OPACITY

   - Use approximations for crude results
   - Interpolate using tables (use approximations outside table bounds)

   ENERGY GENERATION

   - Use formulas for pp, CNO, and 3 α

   THERMODYNAMICS

   - $\Gamma_2 =$ 5/3 for ideal gas, 4/3 for radiation pressure
   - Γ_2 departs from these values when
     - Mixture of ideal gas and radiation
     - Ionization zones

Stellar Models

Steps in constructing a stellar model

2. Use four structure equations to compute r, P, T, L_m vs. m given starting values for these (boundary conditions).

- Given values of $m, r, P, T, L_m, \rho, \kappa, \varepsilon, \Gamma_2$ at shell i
  compute 4 derivatives: $\frac{dr}{dm}, \frac{dP}{dm}, \frac{dL_m}{dm}, \frac{dT}{dm}$
- Use these derivatives to compute r, P, T, L_m at shell $i+1$
  e.g., $r[i+1] = r[i] + \left(\frac{dr}{dm}\right)[i] \times dm$
- Shell size $dm < 10^{-4} M$ near center and surface.

Stellar Models

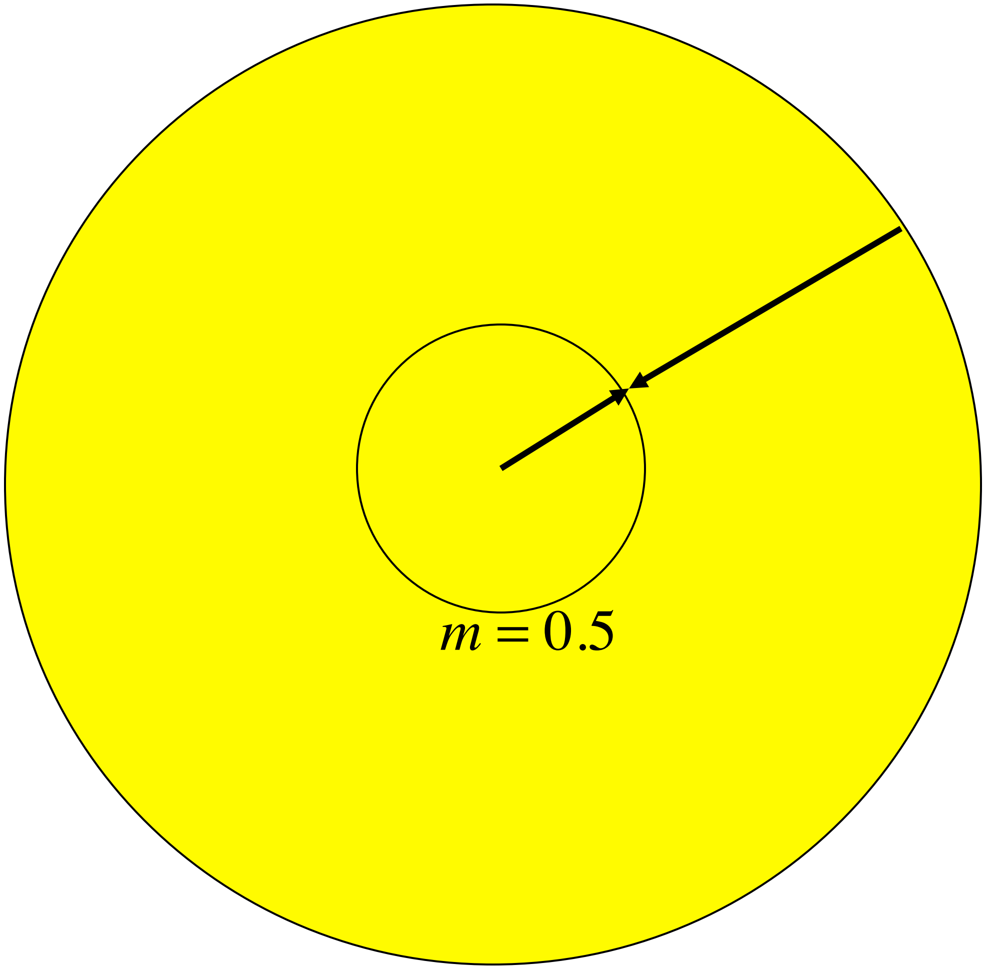
Steps in constructing a stellar model

3. Deal with lack of complete boundary conditions at center or surface.
   - At center, have r, L_m, but not P, T
   - At surface, have P, T, but not r, L_m
   - General approach: guess values for missing conditions at one end, run model, and compare boundary conditions at other end.

     PROBLEM: small changes in conditions at center can cause large differences at surface → difficult to reach convergence.

     SOLUTION: Shoot from both center and surface and meet halfway through star.

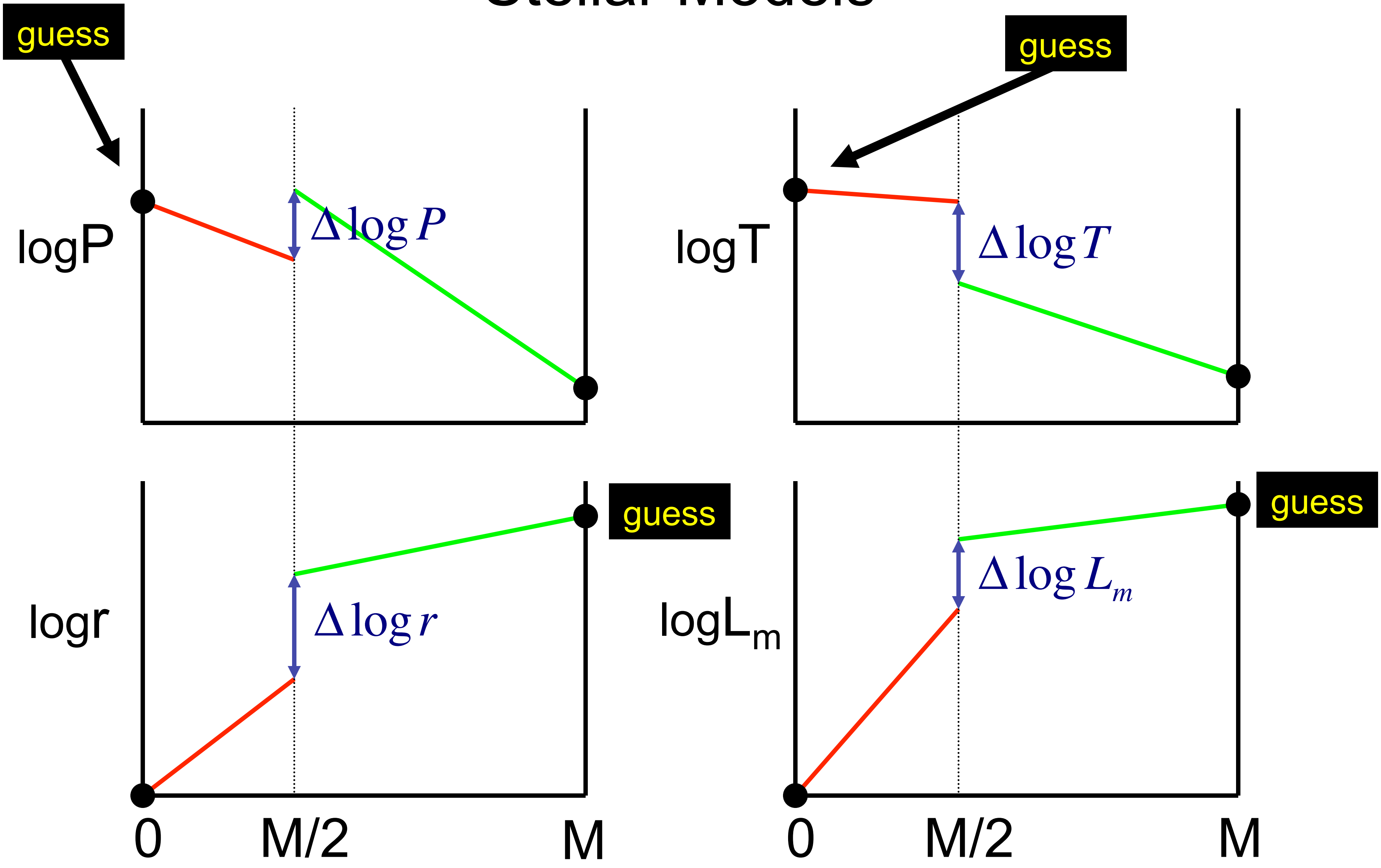
Stellar Models



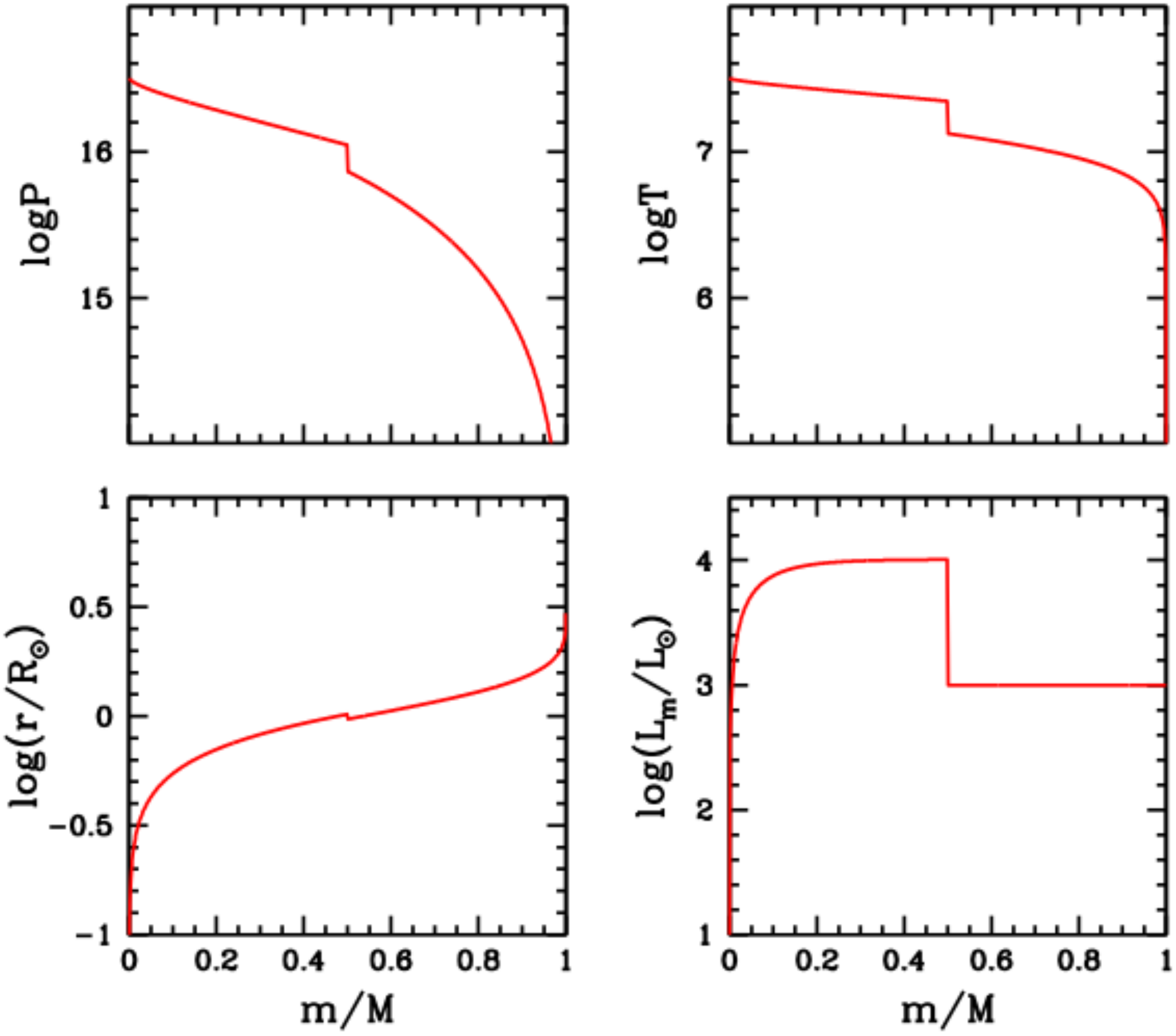
Stellar Models

- Guess values for T_c and P_c and integrate outwards from $m = 0$ to $m = M/2$
- Guess values for R and L and integrate inwards from $m = M$ to $m = M/2$
- Compute discrepancies $\Delta r,\ \Delta P,\ \Delta T,\ \Delta L_m$ at $m = M/2$
- Work in log space: $\log P,\ \log T,\ \log r,\ \log L_m$

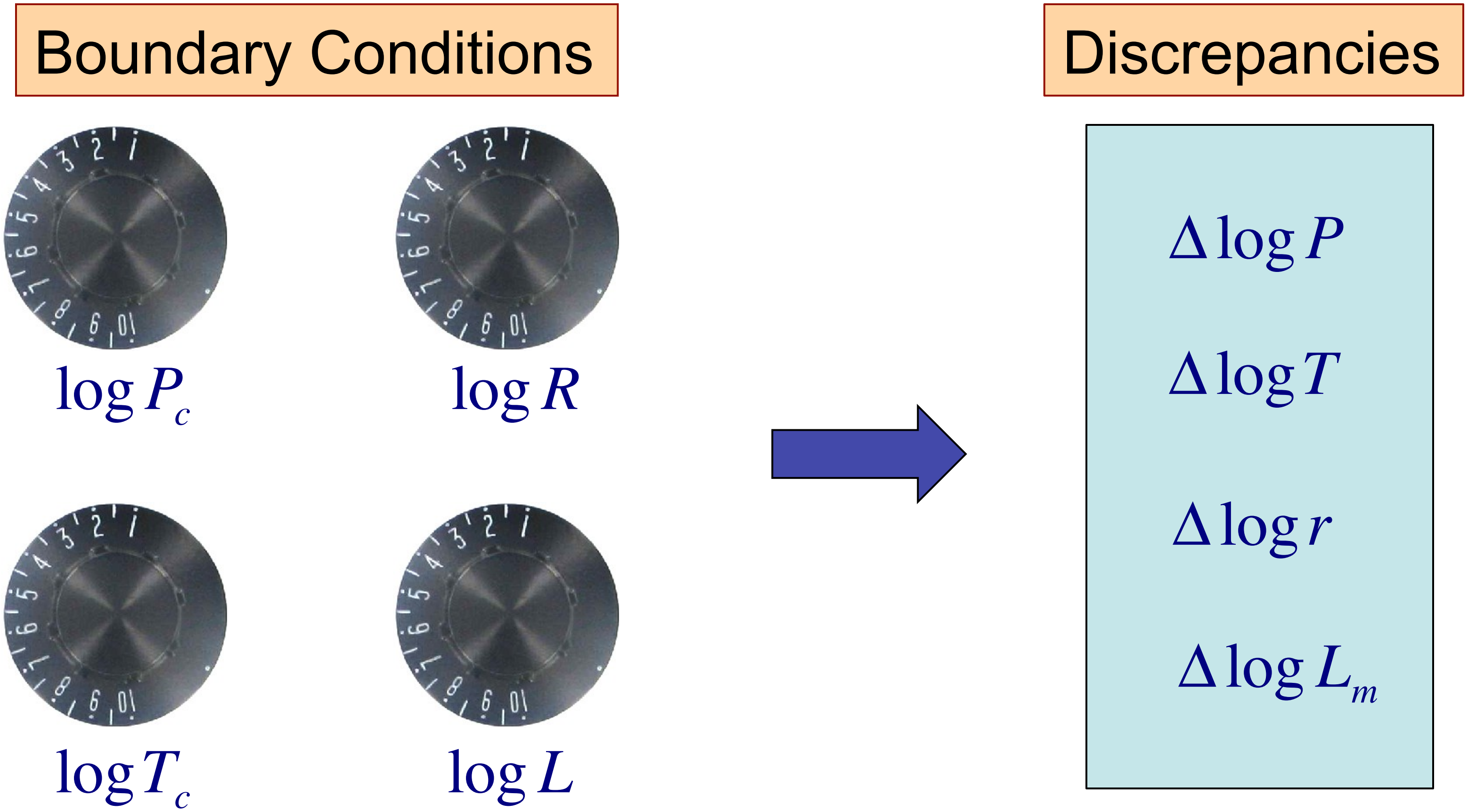
Stellar Models



Stellar Models



Stellar Models



Boundary Conditions

$\log P_c$

$\log R$

$\log T_c$

$\log L$

Discrepancies

$\Delta \log P$

$\Delta \log T$

$\Delta \log r$

$\Delta \log L_m$

Stellar Models

- Repeat using new trial values:

$\left(\log T_c - d\log T_c\right)$ and $\left(\log T_c + d\log T_c\right)$ with $\log P_c$, $\log R$, $\log L$

$\left(\log P_c - d\log P_c\right)$ and $\left(\log P_c + d\log P_c\right)$ with $\log T_c$, $\log R$, $\log L$

$\left(\log R - d\log R\right)$ and $\left(\log R + d\log R\right)$ with $\log P_c$, $\log T_c$, $\log L$

$\left(\log L - d\log L\right)$ and $\left(\log L + d\log L\right)$ with $\log P_c$, $\log T_c$, $\log R$

- Compute new discrepancies in each case.

$$\Delta\log r,\ \Delta\log P,\ \Delta\log T,\ \Delta\log L_m$$

- Get 16 derivatives. e.g.,

$$\frac{\partial(\Delta\log T)}{\partial\log R} = \frac{(\Delta\log T)_{\log R + d\log R} - (\Delta\log T)_{\log R - d\log R}}{2d\log R}$$

Stellar Models

- Use these derivatives to calculate improved boundary conditions.

$$\begin{aligned}\log P_c' &= \log P_c + \delta \log P_c \\ \log T_c' &= \log T_c + \delta \log T_c \\ \log R' &= \log R + \delta \log R \\ \log L' &= \log L + \delta \log L\end{aligned}$$

- General idea:

$$-\Delta \log r = \delta \log P_c \frac{\partial(\Delta \log r)}{\partial \log P_c} \rightarrow \delta \log P_c = -\Delta \log r \left[\frac{\partial(\Delta \log r)}{\partial \log P_c} \right]^{-1}$$

- More complicated with 4 variables!

Stellar Models

$$\begin{pmatrix} \frac{\partial(\Delta\log P)}{\partial\log P_c} & \frac{\partial(\Delta\log P)}{\partial\log T_c} & \frac{\partial(\Delta\log P)}{\partial\log R} & \frac{\partial(\Delta\log P)}{\partial\log L} \\ \frac{\partial(\Delta\log T)}{\partial\log P_c} & \frac{\partial(\Delta\log T)}{\partial\log T_c} & \frac{\partial(\Delta\log T)}{\partial\log R} & \frac{\partial(\Delta\log T)}{\partial\log L} \\ \frac{\partial(\Delta\log r)}{\partial\log P_c} & \frac{\partial(\Delta\log r)}{\partial\log T_c} & \frac{\partial(\Delta\log r)}{\partial\log R} & \frac{\partial(\Delta\log r)}{\partial\log L} \\ \frac{\partial(\Delta\log L_m)}{\partial\log P_c} & \frac{\partial(\Delta\log L_m)}{\partial\log T_c} & \frac{\partial(\Delta\log L_m)}{\partial\log R} & \frac{\partial(\Delta\log L_m)}{\partial\log L} \end{pmatrix} \times \begin{pmatrix} \delta\log P_c \\ \delta\log T_c \\ \delta\log R \\ \delta\log L \end{pmatrix} = \begin{pmatrix} \Delta\log P \\ \Delta\log T \\ \Delta\log r \\ \Delta\log L_m \end{pmatrix}$$

- Invert matrix to solve for $\delta\log P_c,\ \delta\log T_c,\ \delta\log R,\ \delta\log L$

  (actually take smaller steps ~0.1x)

- Iterate until convergence is reached: discrepancies vanish (i.e., drop below a threshold value)

Gravitational collapse of gas

- Assume a gas cloud of mass M and diameter D

- Sound speed for ideal gas is $c_s = \sqrt{\gamma \frac{P}{\rho}} = \sqrt{\gamma \frac{nkT}{\rho}} = \sqrt{\gamma \frac{kT}{m}}$

- Time for sound wave to cross the cloud $t_{sound} = \frac{D}{c_s} == D\left(\frac{m}{\gamma kT}\right)^{1/2}$

- Time for free-fall collapse is $t_{ff} = \frac{1}{\sqrt{G\rho}}$

- Gravity beats pressure support when $t_{ff} < t_{sound}$

Gravitational collapse of gas

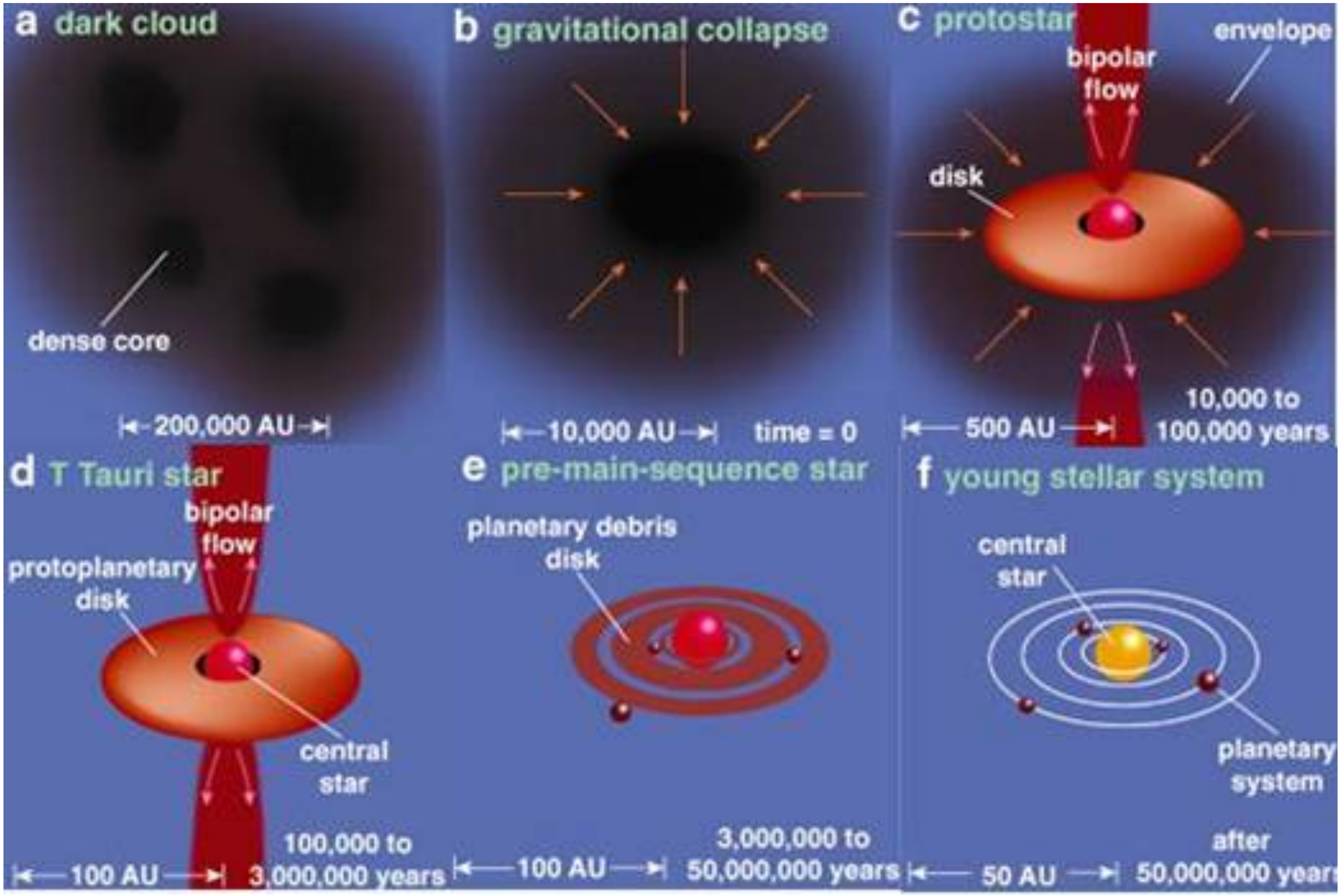
- Critical cloud size is then $t_{ff} = t_{sound} \rightarrow \frac{1}{\sqrt{G\rho}} = D\left(\frac{m}{\gamma kT}\right)^{1/2}$

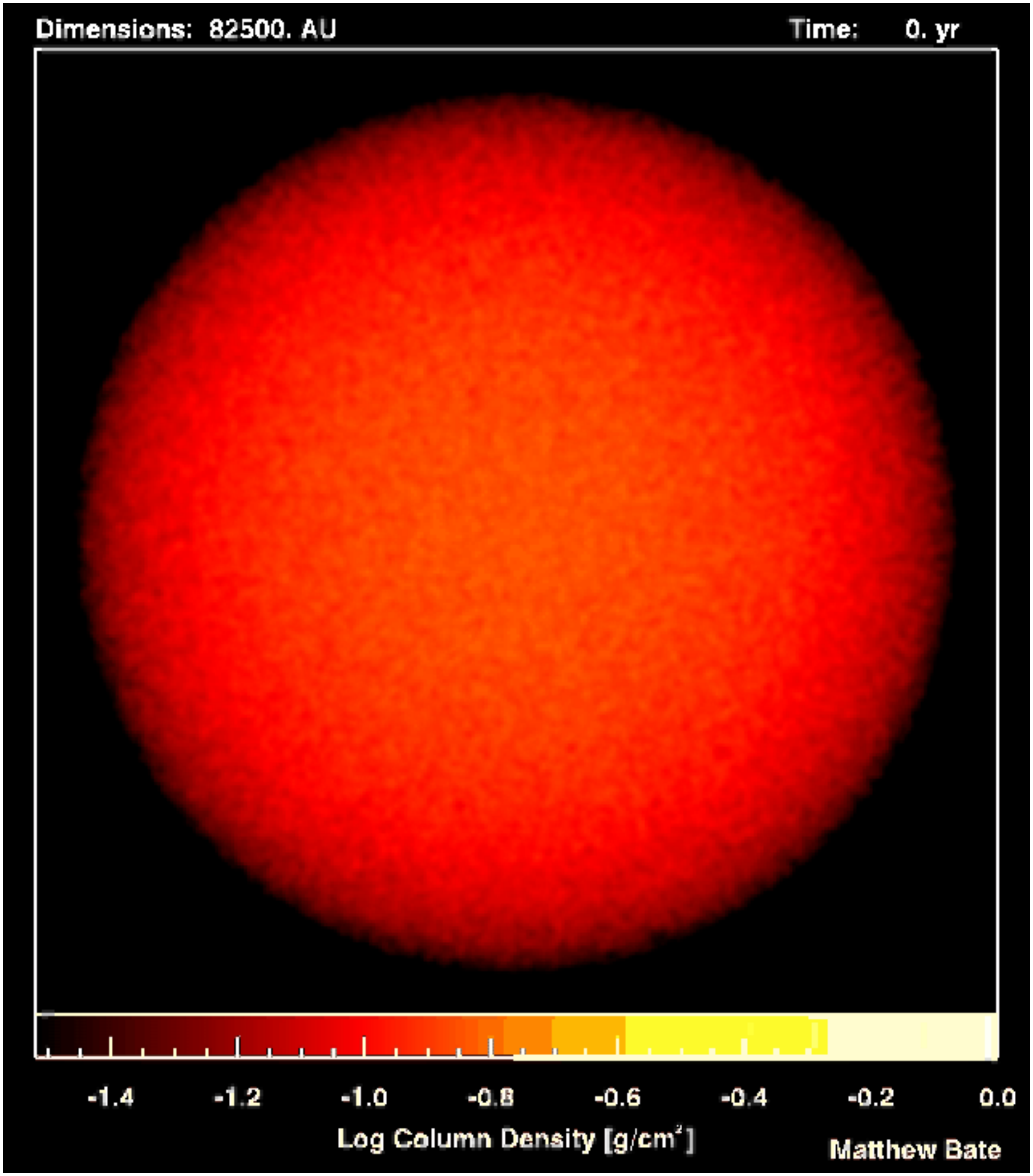
This is the Jeans length $\lambda_J = \left(\frac{\gamma kT}{mG\rho}\right)^{1/2}$

- Associated Jeans mass is $M_J = \frac{4}{3}\pi\left(\frac{\lambda_J}{2}\right)^3 \rho \rightarrow$

$$M_J = \frac{\pi}{6}\left(\frac{\gamma kT}{mG}\right)^{3/2} \rho^{5/2}$$

Star Formation





Starting Inputs:

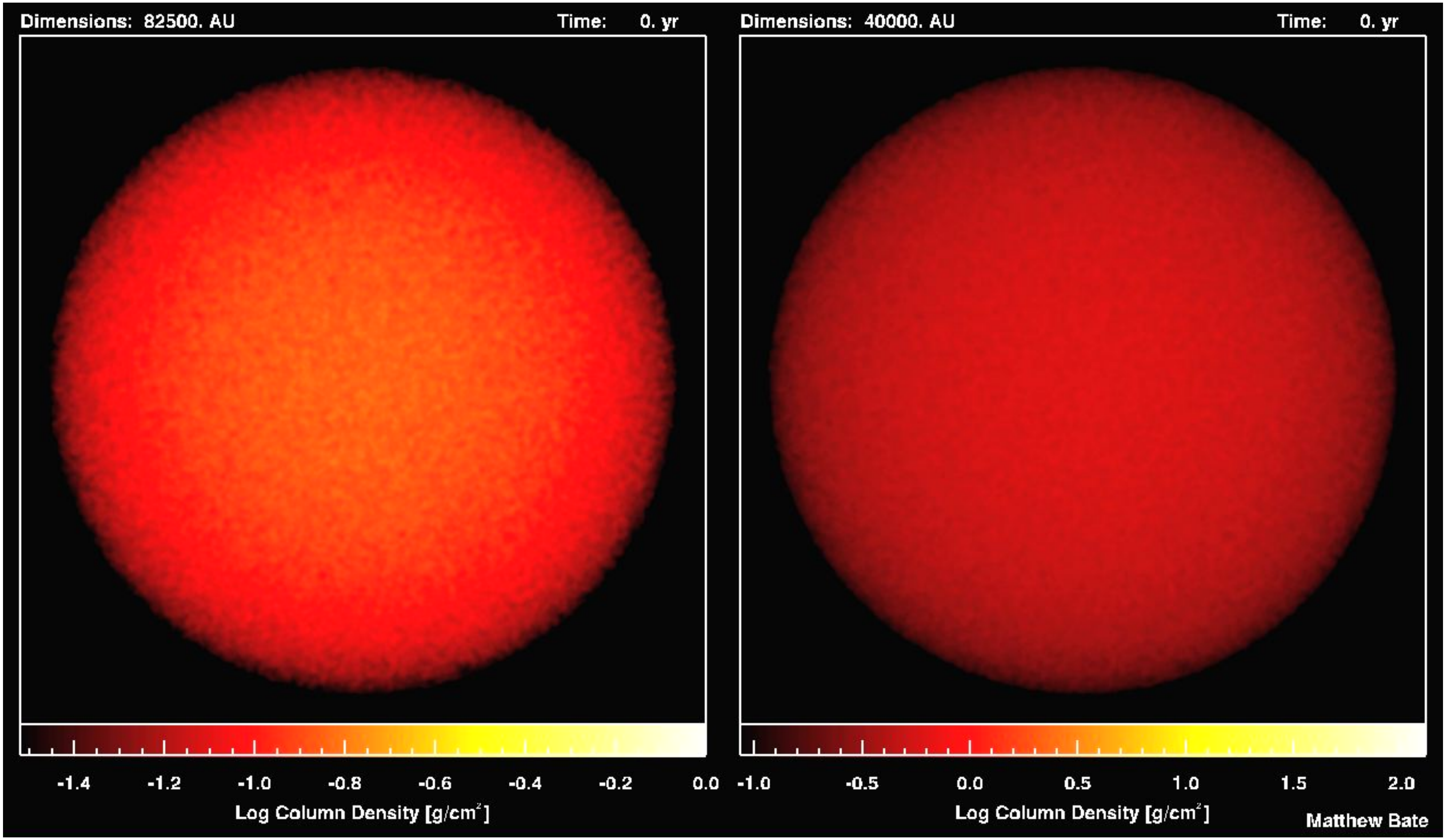
Mass: 50 Msun
Diameter: 0.375 pc
Temperature: 10 K
Mean mol. Weight: 2.46
(Jeans mass = 1 Msun)
Time evolved = 266K years

Initial density and turbulence spectra.

Computing:

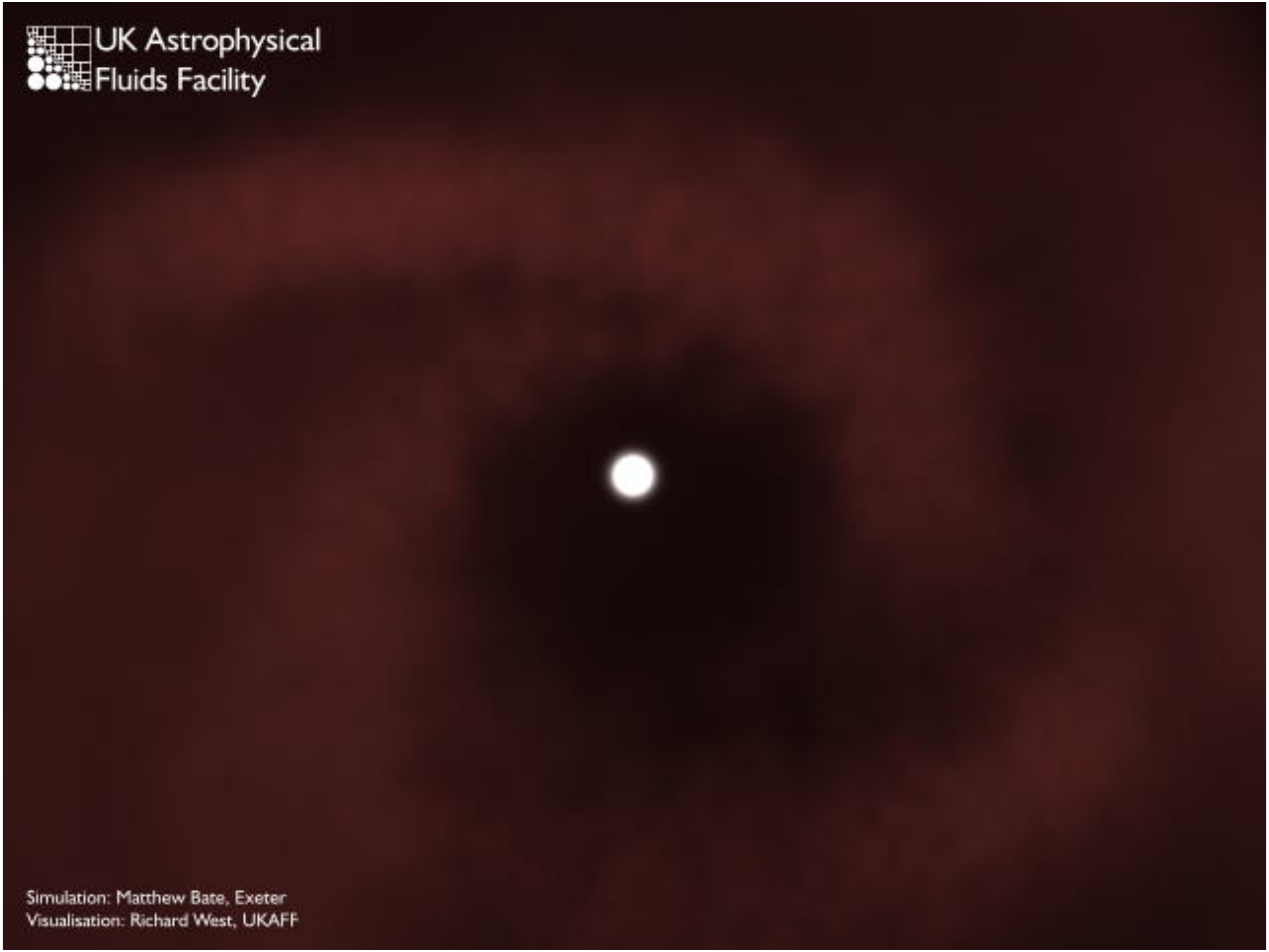
SPH code, 3.5 M particles
100K CPU-hours on 64 CPUs
(65 days)
Resolution: 1-5 AU

Bate, M. R., et al. (2002)



Original cloud

Denser cloud



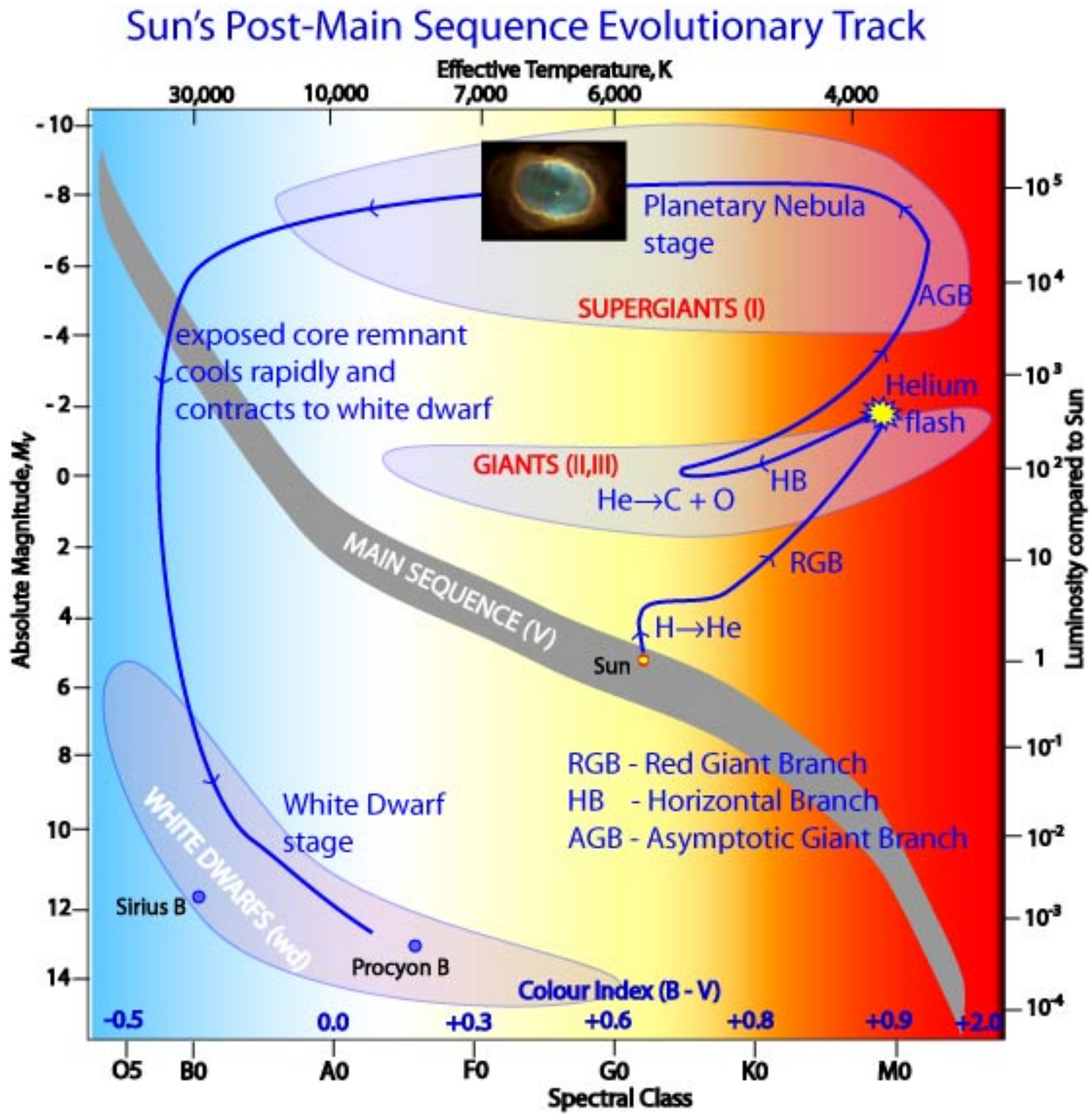
Simulation: Matthew Bate, Exeter
Visualisation: Richard West, UKAFF

Star Formation

These simulations show:

- Star formation is a very chaotic and dynamic process
- Stars form so close together that they often interact before growing to full size
- Young stars compete for remaining gas with more massive stars
- About half the objects are kicked out of the cluster before they can grow enough to start fusion : brown dwarfs
- Many of the encounters btw. young stars and brown dwarfs strip the dusty disks off the stars suggesting planetary systems could be rare

Evolution of the Sun



Stages in Evolution:

Hayashi track

Deuterium burning

Main Sequence

H → He in core

Red Giant Branch

He core, H → He in shell

Tip of the Red Giant Branch

Degenerate He core → He flash

Horizontal Branch

He → C,O in core, H → He in shell

Asymptotic Giant Branch

C,O core, He → C,O and H → He in shells

Planetary Nebula

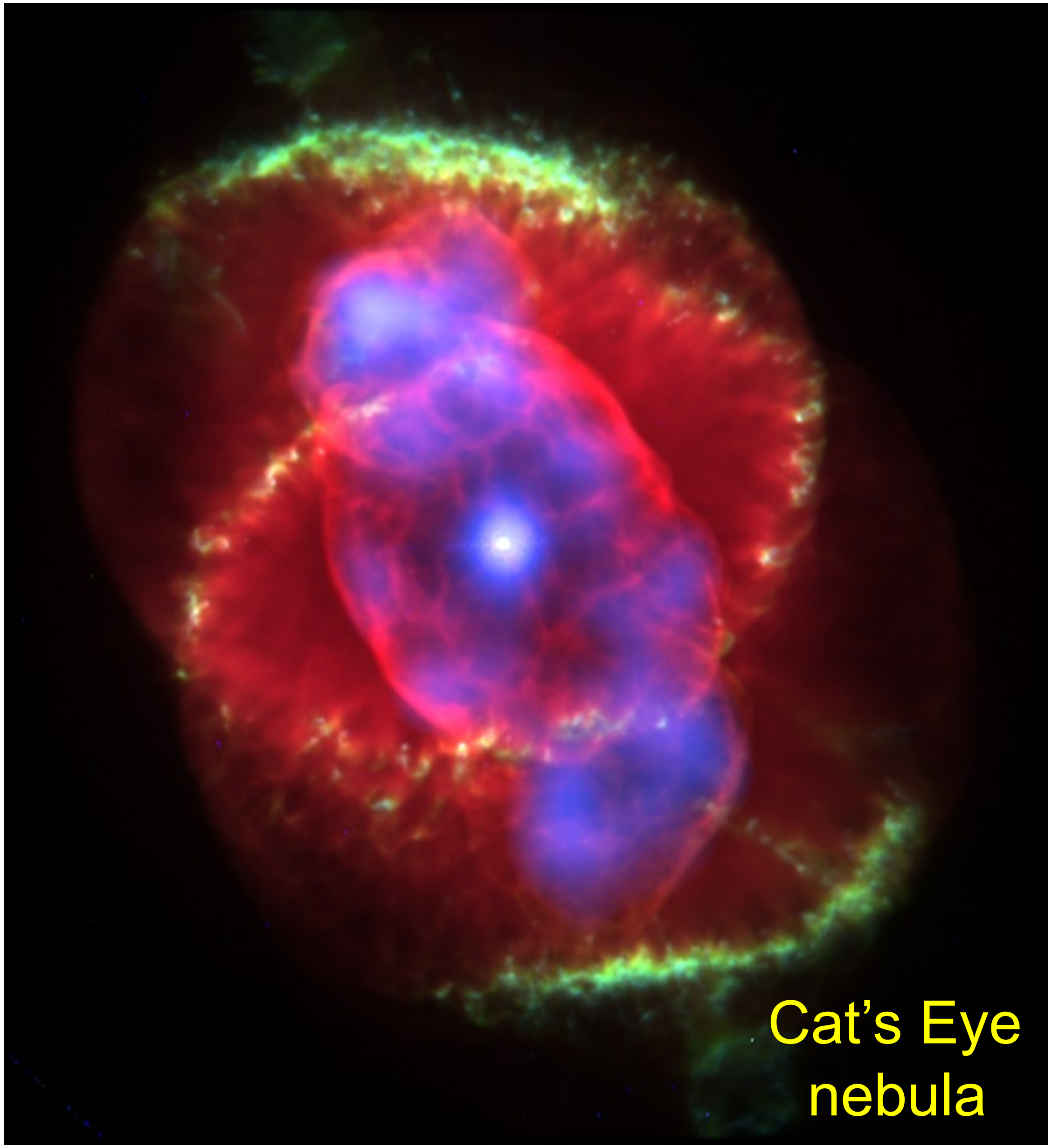
Not massive enough to burn C,O

Sheds outer layers.

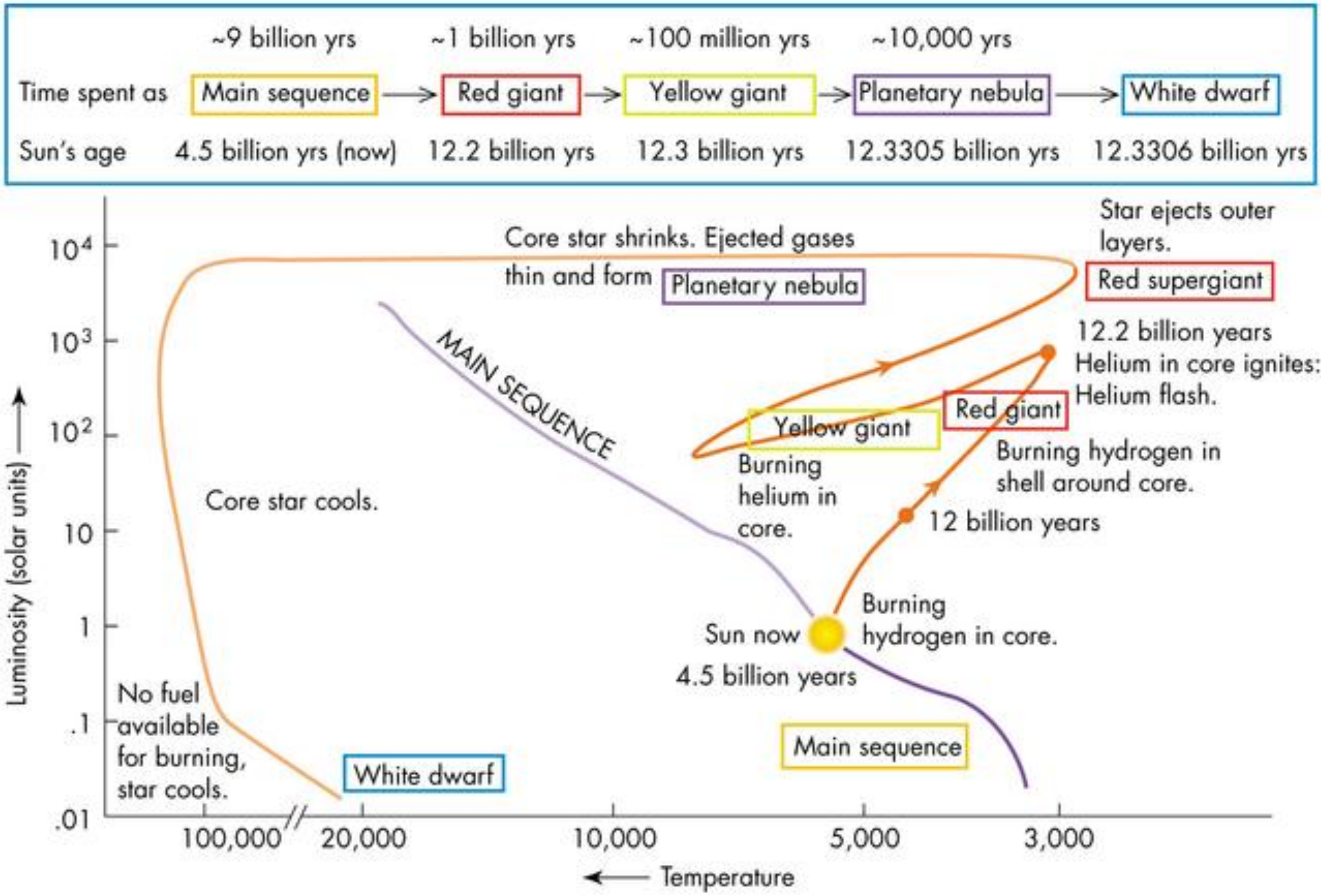
White Dwarf

Degenerate C,O

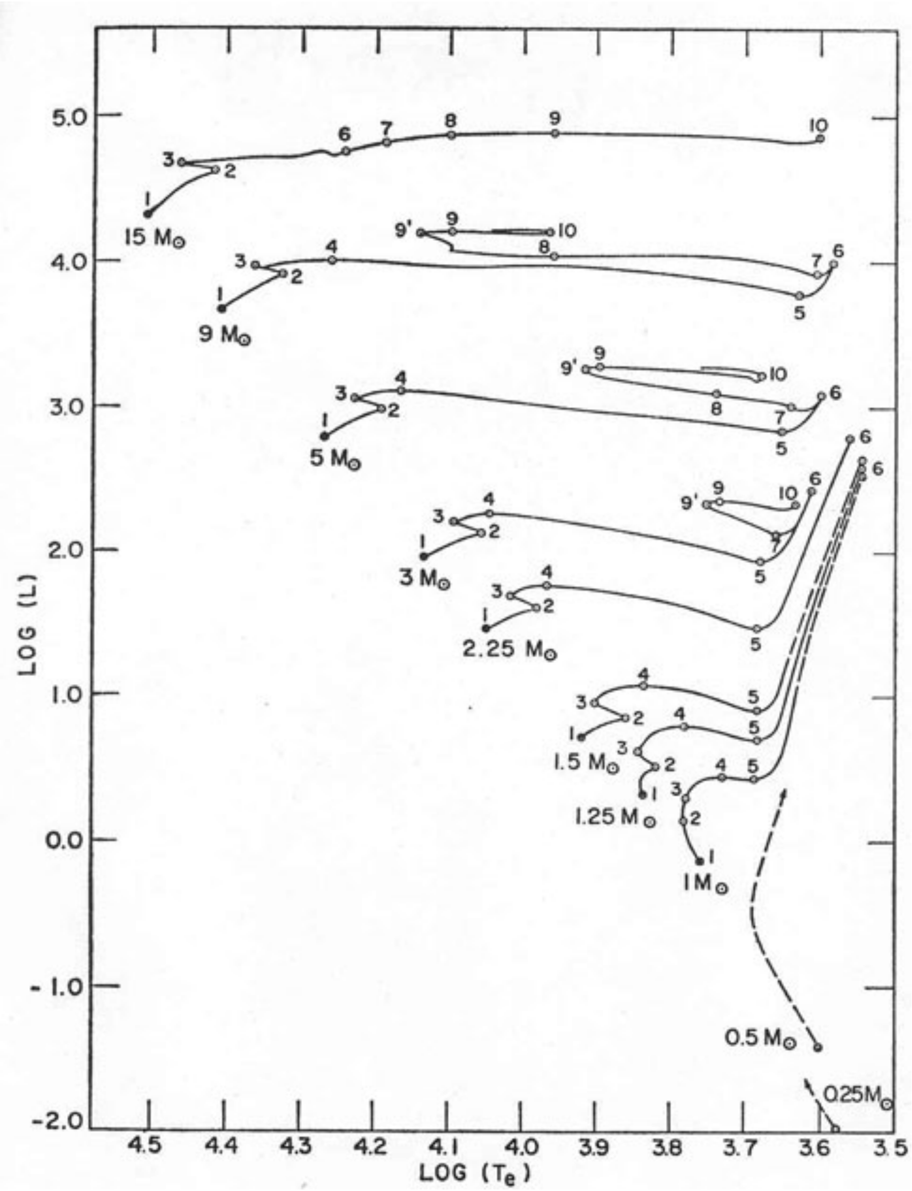
Planetary Nebulae



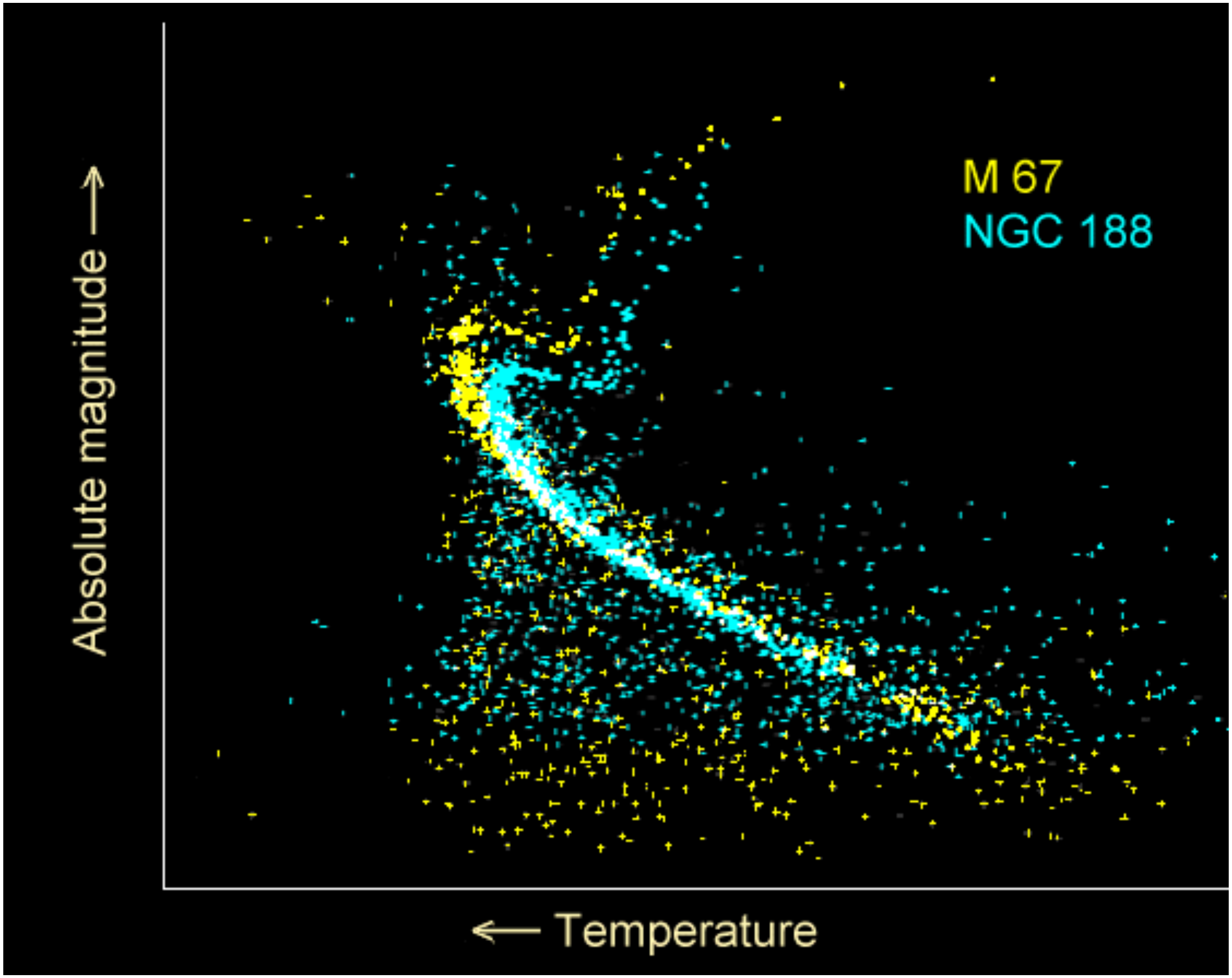
Cat’s Eye nebula


~9 billion yrs
~1 billion yrs
~100 million yrs
~10,000 yrs
Time spent as
Main sequence
Red giant
Yellow giant
Planetary nebula
White dwarf
Sun's age
4.5 billion yrs (now)
12.2 billion yrs
12.3 billion yrs
12.3305 billion yrs
12.3306 billion yrs
Star ejects outer layers.
Core star shrinks. Ejected gases thin and form
Planetary nebula
Red supergiant
12.2 billion years
Helium in core ignites:
Helium flash.
MAIN SEQUENCE
Yellow giant
Red giant
Burning hydrogen in shell around core.
Burning helium in core.
Core star cools.
12 billion years
Burning hydrogen in core.
Sun now
4.5 billion years
No fuel available for burning, star cools.
White dwarf
Main sequence
Luminosity (solar units)
10^4
10^3
10^2
10
1
.1
.01
100,000
20,000
10,000
5,000
3,000
Temperature

Post - Main Sequence Evolution

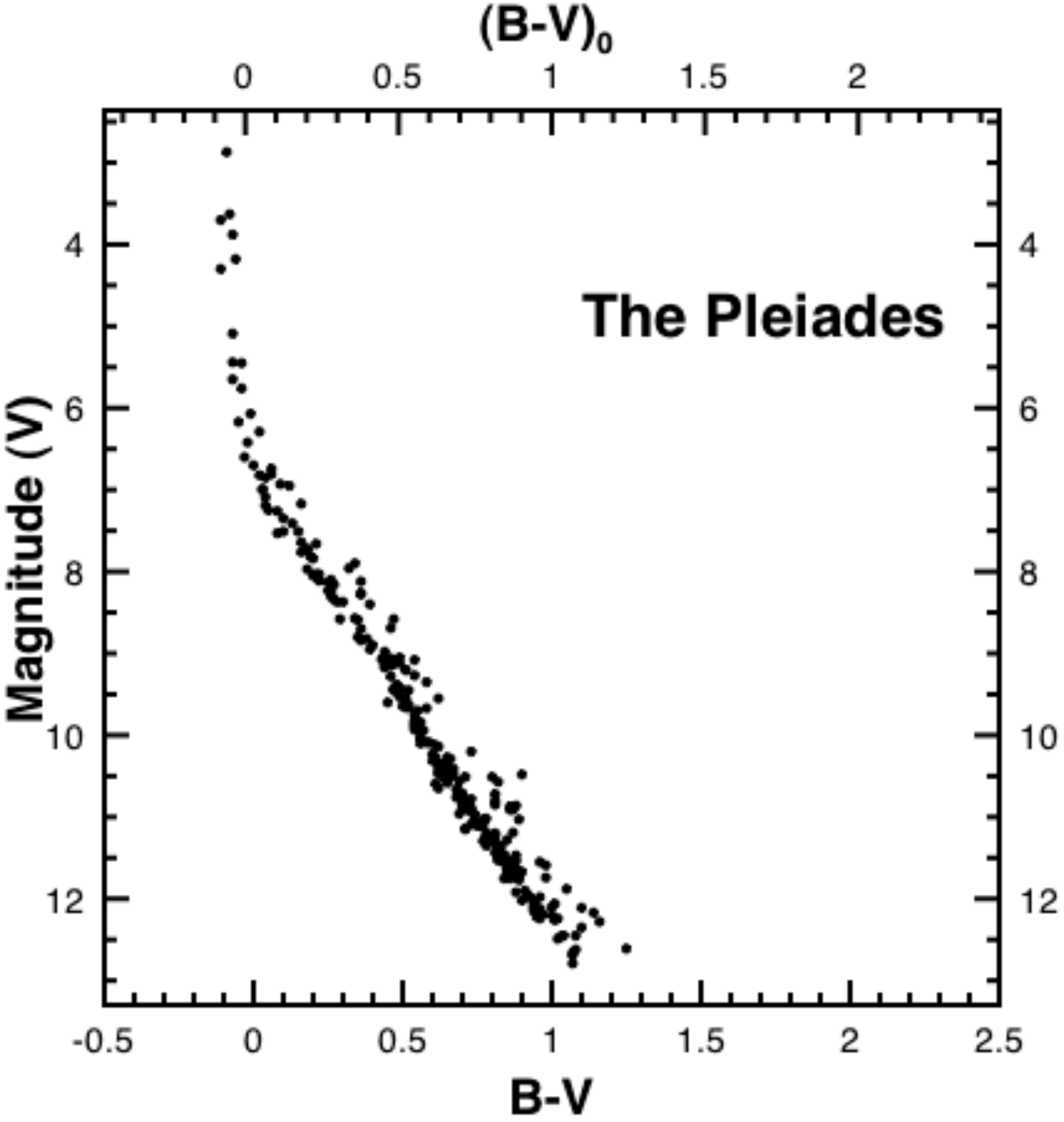


Main Sequence Turn-off

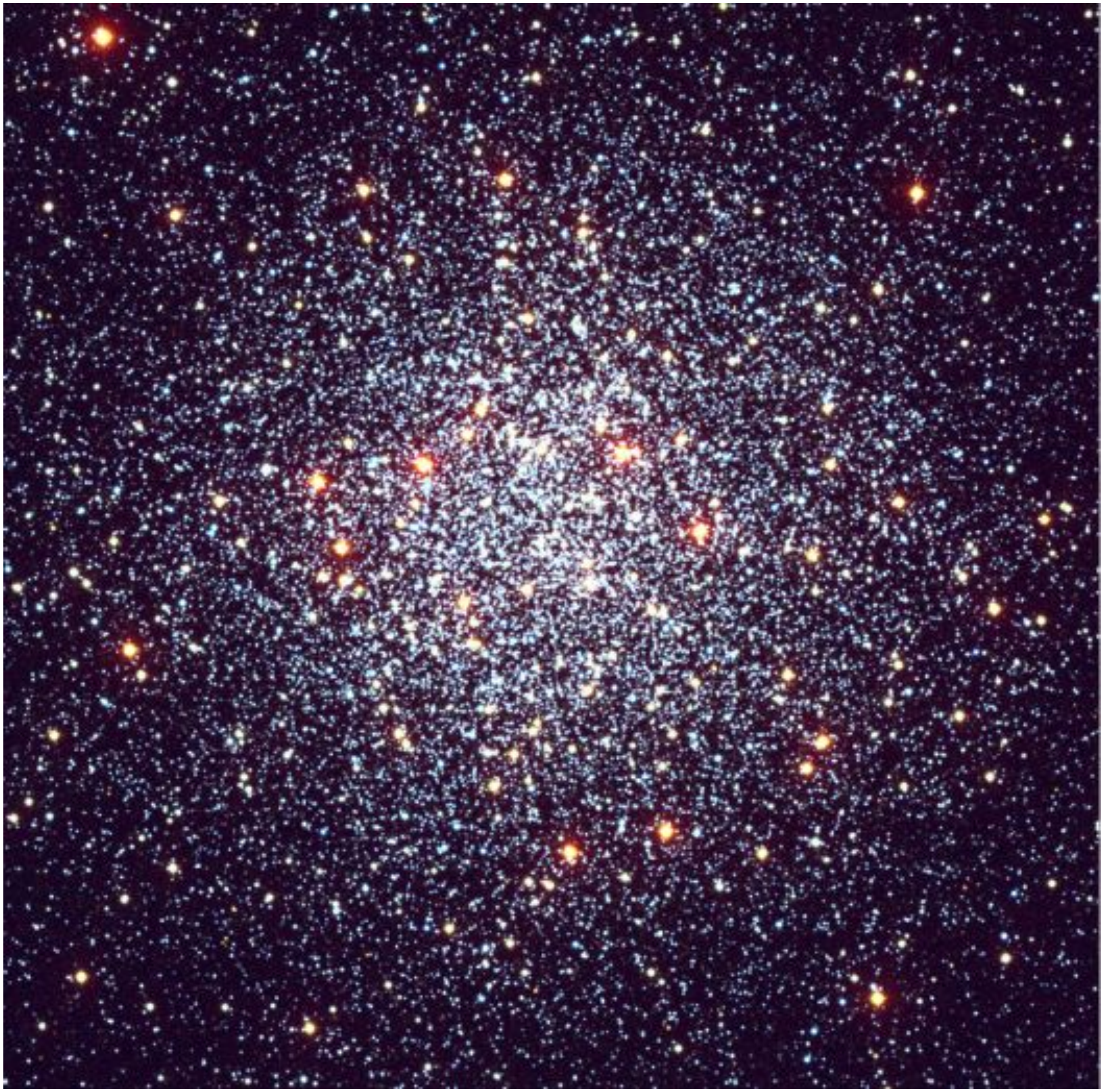


Pleiades Open Cluster

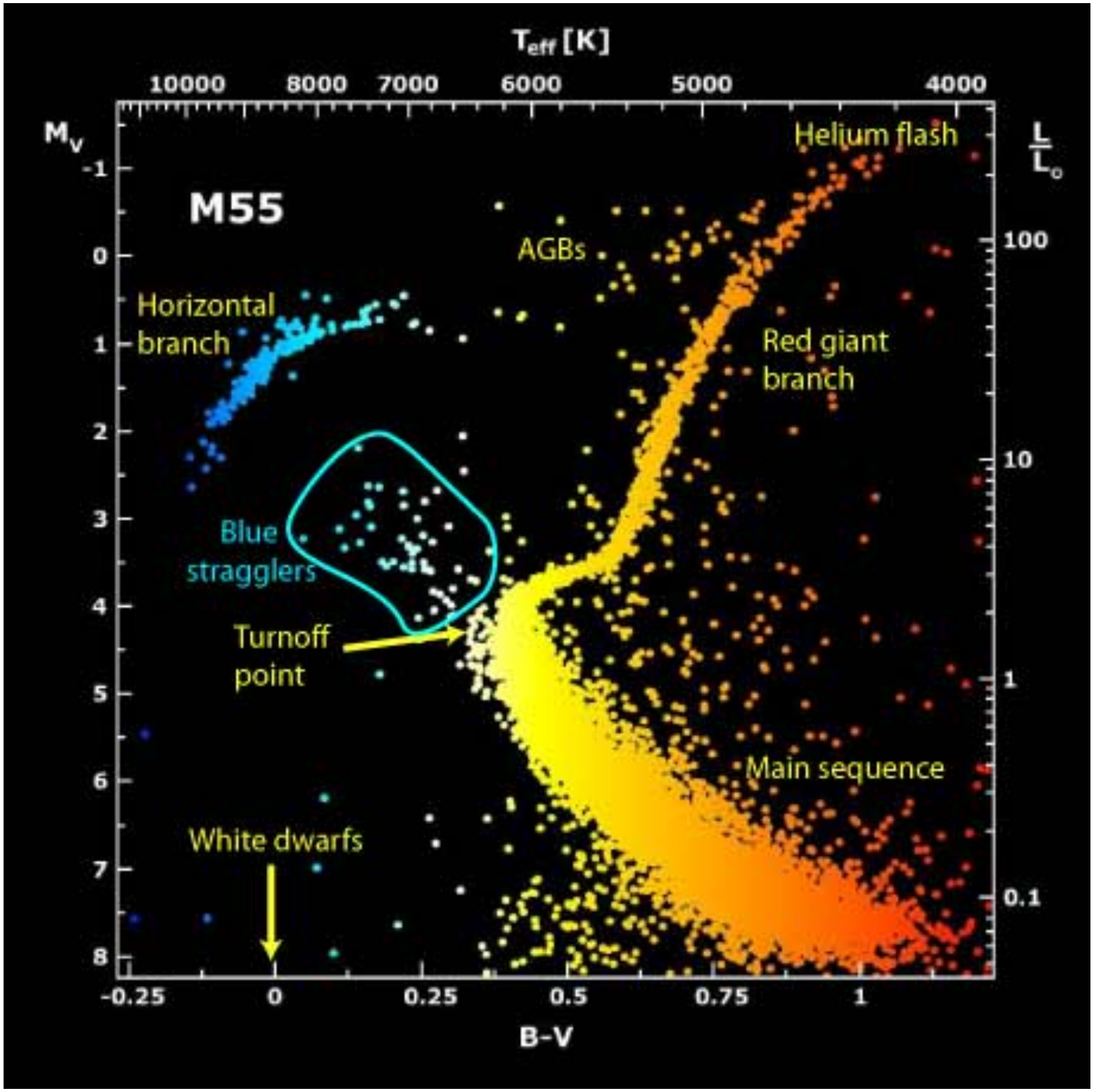
Pleiades Open Cluster

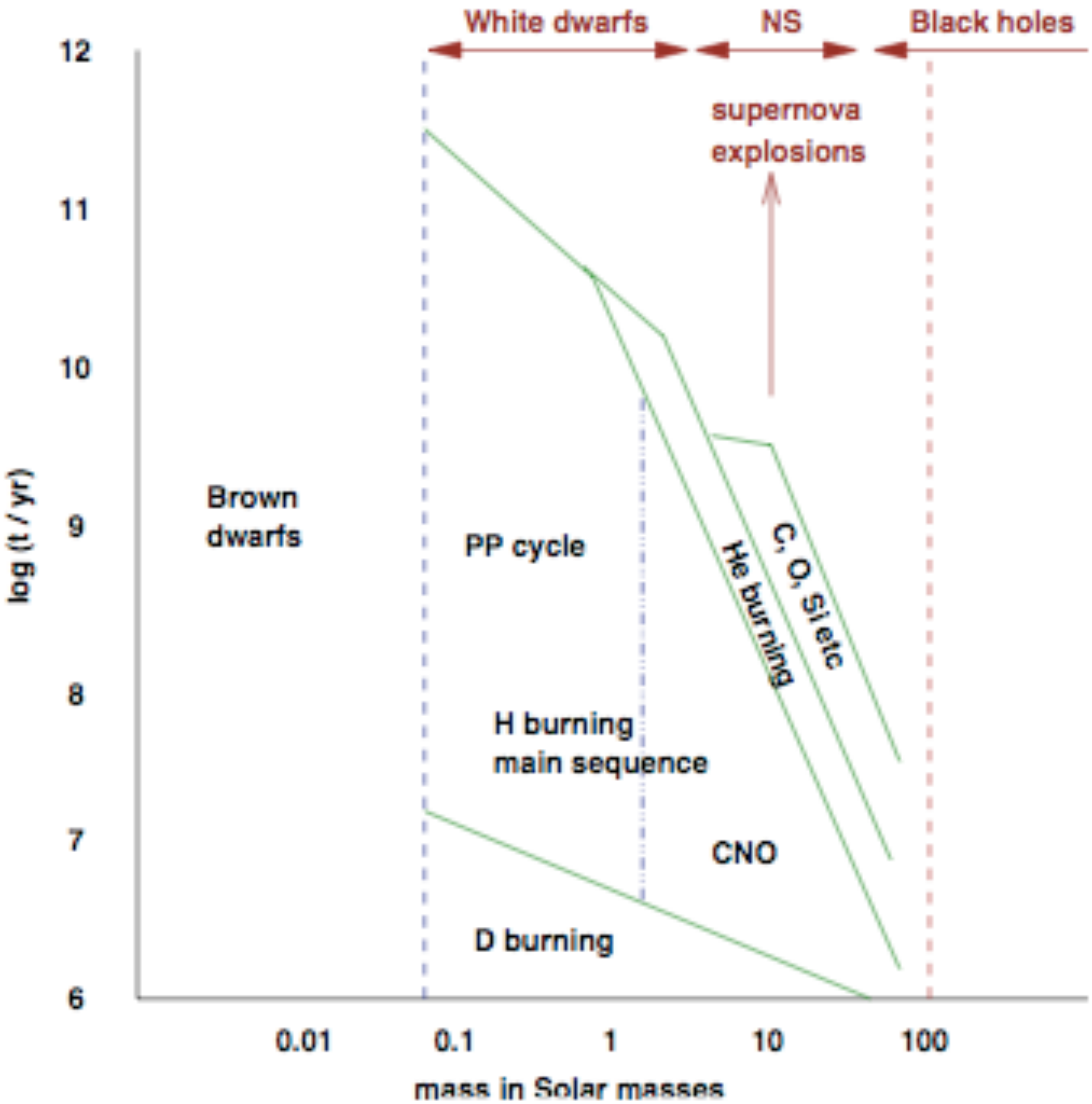


M55 Globular Cluster



M55 Globular Cluster





Post-main-sequence evolution

MASS	FUSION	REMNANT
$M < 0.08 M_\odot$	No fusion	Brown Dwarf
$0.08 M_\odot < M < 0.5 M_\odot$	Central H burning Formation of degenerate core No He burning	He White Dwarf
$0.5 M_\odot < M < 2 M_\odot$	Central H burning Helium flash	CO White Dwarf
$2 M_\odot < M < 8 M_\odot$	Central H burning He ignites in non-degenerate core	CO White Dwarf
$8 M_\odot < M < 20 M_\odot$	Numerous burning phases Type II supernova	Neutron Star
$20 M_\odot < M < 50 M_\odot$	Numerous burning phases Type II supernova	Black Hole
$M > 50 M_\odot$	Numerous burning phases Hypernova/Collapsar	Black Hole