Convection

- If luminosity is transported by radiation, then it must obey

$$L_r = -\frac{16\pi a c r^2 T^3}{3\rho\kappa_R}\frac{dT}{dr}$$

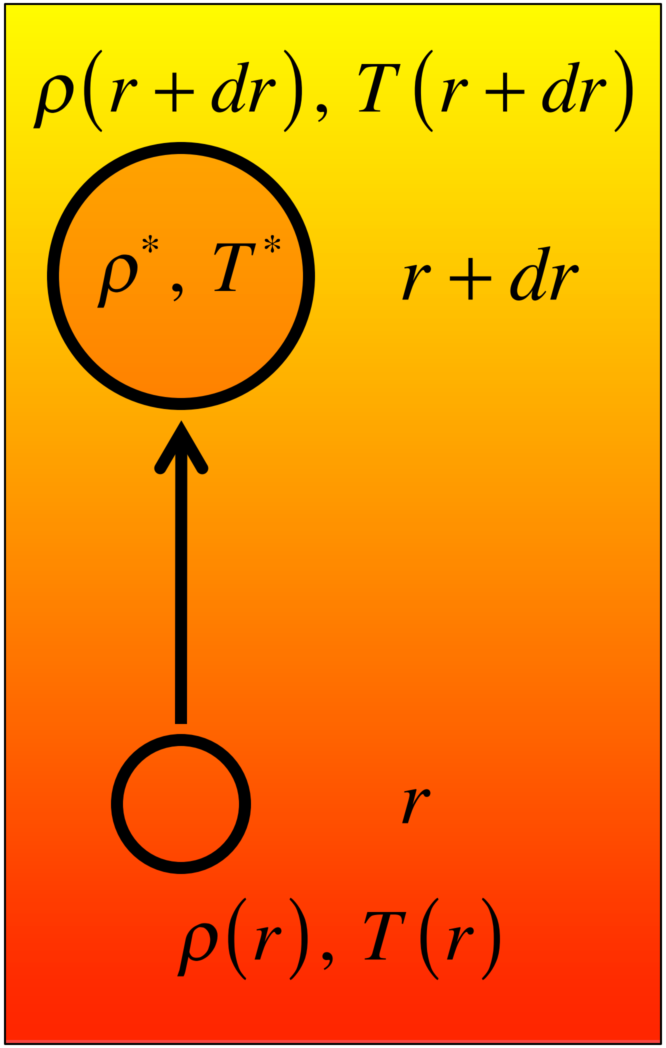
- In a steady state, the energy transported per time at radius *r* must be equal to the energy generation rate in the stellar interior.

$$\rightarrow \frac{dT}{dr} \text{ will be large if } \begin{cases} L \text{ is large} \\ \kappa_R \text{ is large} \end{cases}$$

- The temperature gradient cannot be arbitrarily large. If it gets too steep → convection takes over as the main mode of energy transport.

Convection: Stability Criteria

- Uniform composition, $T(r)$, $\rho(r)$ profiles
- Displace a mass element by *dr* without exchanging heat with the environment (adiabatically).
- Element expands to maintain pressure balance with its environment.
- Its new density ρ^* and temperature T^* will not in general equal the ambient values at $r+dr$.



Convection: Stability Criteria

- Since the change is adiabatic:

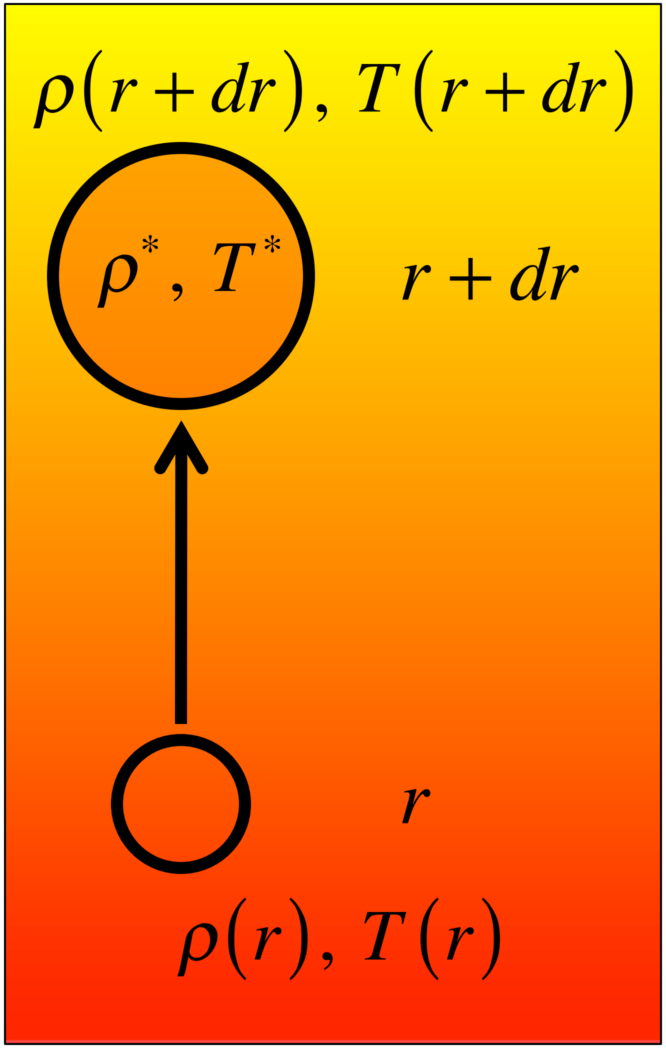
$$\left(\frac{d\ln P}{d\ln\rho}\right)_{\rm ad} = \Gamma_1 \quad \rightarrow \frac{\rho}{P}\left(\frac{dP}{d\rho}\right)_{\rm ad} = \Gamma_1$$

$$\rightarrow (d\rho)_{\rm ad} = \frac{\rho}{\Gamma_1 P}(dP)_{\rm ad}$$

$$\rho^* = \rho(r) + (d\rho)_{\rm ad} \quad = \rho(r) + \frac{\rho}{\Gamma_1 P}(dP)_{\rm ad}$$

$$= \rho(r) + \frac{\rho}{\Gamma_1 P}\left(\frac{dP}{dr}\right)_{\rm ad} dr$$

- Since pressure equilibrium applies: $$\left(\frac{dP}{dr}\right)_{\rm ad} = \frac{dP}{dr}$$



Convection: Stability Criteria

- If $\rho^* > \rho(r+dr)$: displaced element will be denser than its surroundings and will settle back down → STABILITY

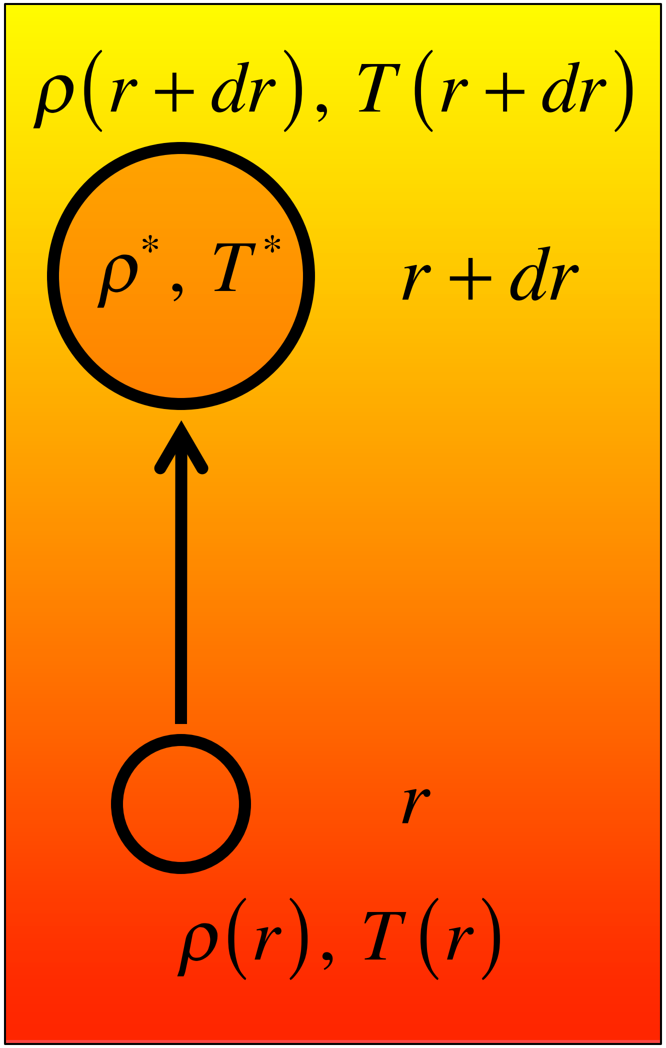
- If $\rho^* < \rho(r+dr)$: buoyancy will cause element to rise further → INSTABILITY

- Stability criterion is:

$$\rho(r) + \frac{\rho}{\Gamma_1 P}\frac{dP}{dr}dr > \rho(r+dr)$$

$$\rightarrow \frac{\rho(r+dr) - \rho(r)}{dr} < \frac{\rho}{\Gamma_1 P}\frac{dP}{dr}$$

$$\boxed{\frac{d\rho}{dr} < \frac{\rho}{\Gamma_1 P}\frac{dP}{dr}}$$



Convection: Stability Criteria

- Since the change is adiabatic:

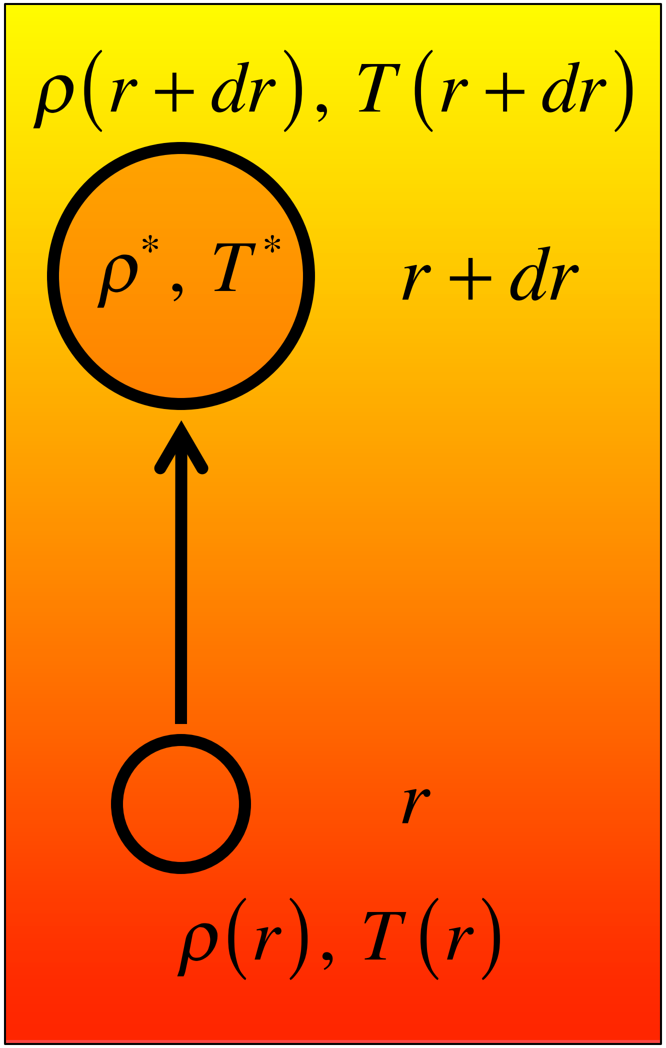
$$\left(\frac{d\ln P}{d\ln T}\right)_{\text{ad}} = \frac{\Gamma_2}{\Gamma_2 - 1} \rightarrow \frac{T}{P}\left(\frac{dP}{dT}\right)_{\text{ad}} = \frac{\Gamma_2}{\Gamma_2 - 1}$$

$$\rightarrow (dT)_{\text{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{P}(dP)_{\text{ad}}$$

$$T^* = T(r) + (dT)_{\text{ad}}$$

$$= T(r) + \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}(dP)_{\text{ad}}$$

$$= T(r) + \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}dr$$



Convection: Stability Criteria

- If $T^* < T(r+dr)$: displaced element will be cooler than its surroundings and will settle back down → STABILITY

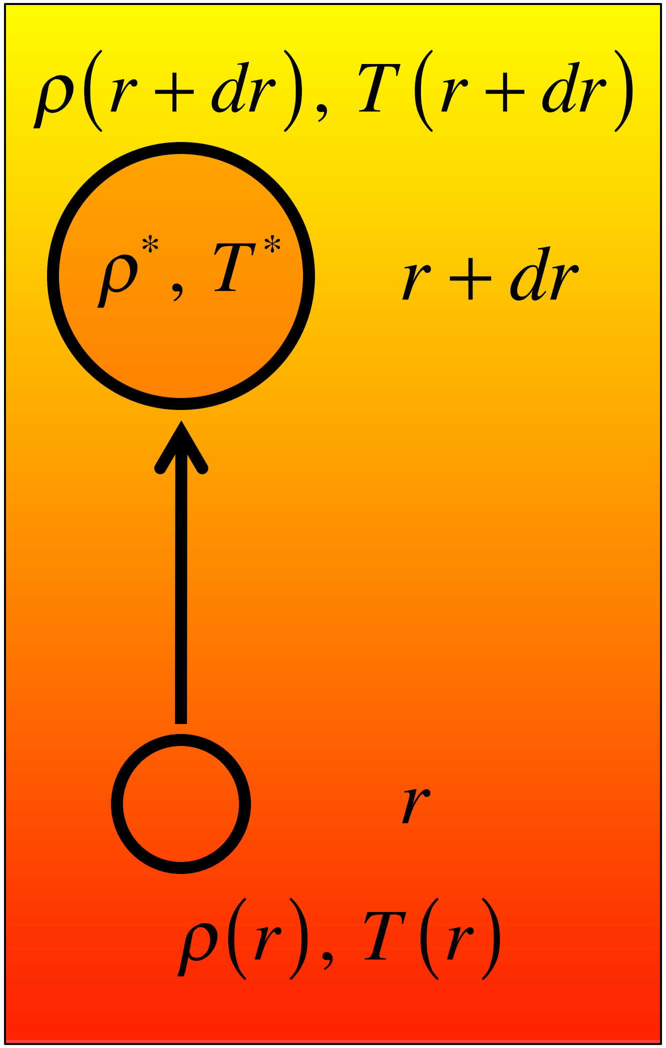
- If $T^* > T(r+dr)$: element will be hotter and thus rise → INSTABILITY

- Stability criterion is:

$$T(r) + \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}dr < T(r+dr)$$

$$\rightarrow \frac{T(r+dr) - T(r)}{dr} > \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\boxed{\frac{dT}{dr} > \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}}$$

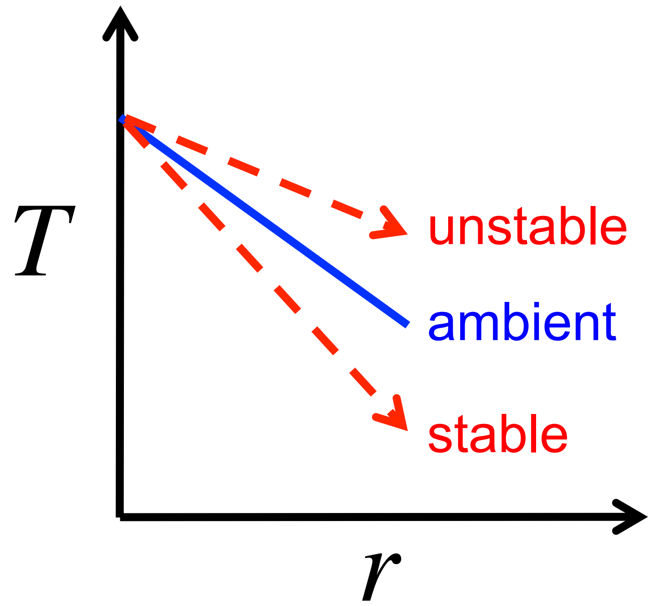


Convection: Stability Criteria

- Stability criterion is:

$$\frac{dT}{dr} > \left(\frac{dT}{dr}\right)_{\mathrm{ad}}$$

$$\left|\frac{dT}{dr}\right| < \left|\left(\frac{dT}{dr}\right)_{\mathrm{ad}}\right|$$



- Too rapid a change in temperature → CONVECTION

$$\frac{dT}{dr} < \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

Convection: Stability Criteria

- Convert to maximum luminosity that can be carried by radiation

$$\frac{dT}{dr} = -\frac{3\rho\kappa_R L_r}{16\pi a c r^2 T^3} \qquad > \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\frac{3\rho\kappa_R L_r}{16\pi a c r^2 T^3} < -\left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2}$$

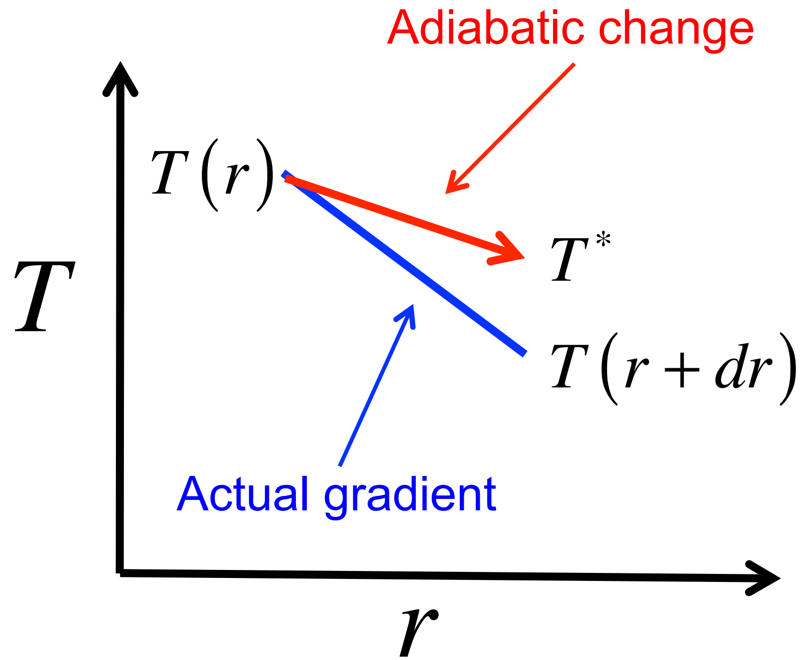
$$L_r < \frac{16\pi a c G}{3\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

- If L exceeds this value → CONVECTION

Convection: Stability Criteria

- In an unstable region, displaced element is hotter than ambient gas
  → continues to rise

- Eventually, radiation will leak out of the rising gas element.
  → extra energy flux from hotter to cooler regions



- What causes convective instability in some regions of a star?

$$L_{\max} \sim \frac{1}{\kappa_R}\left(1-\frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

When L_{max} is low → convection

Convection: Stability Criteria

Near surface

$$m \approx M$$

$$\frac{T^4}{P} \approx \text{const}$$

$$L_{max} \sim \frac{1}{\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

$$L_{max} \sim \frac{1}{\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)$$

Remember

$$\Gamma_2 = 5/3$$

Vulnerable to convection if

- κ_R is large — Occurs when there is a large contribution from atomic processes.
- $\Gamma_2 \rightarrow 1$ — Occurs in ionization zones where Γ_2 dips below 4/3

So, atomic processes + ionization zones → convection will happen
In regions where T~10^4-10^5K and H is being ionized.

Not in high mass stars

Convection: Stability Criteria

Near center

$$L_{\max} \sim \frac{1}{\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

κ_R is low

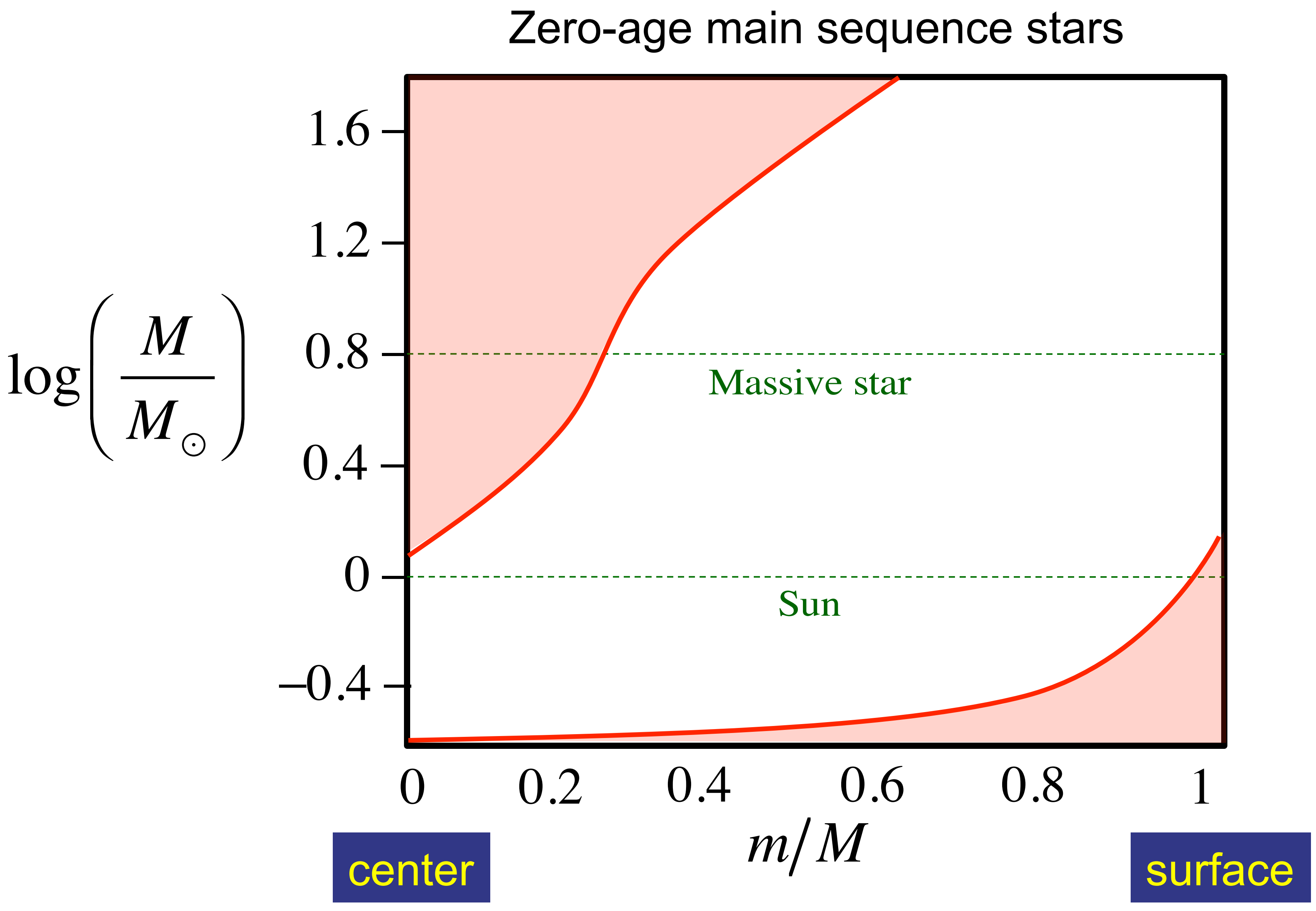
Γ_2 is high

$$L_{\max} \sim m$$

Vulnerable to convection if there is a large luminosity at low mass.

This happens when nuclear energy generation is a very strong function of *T*. CNO in massive stars.

Convection

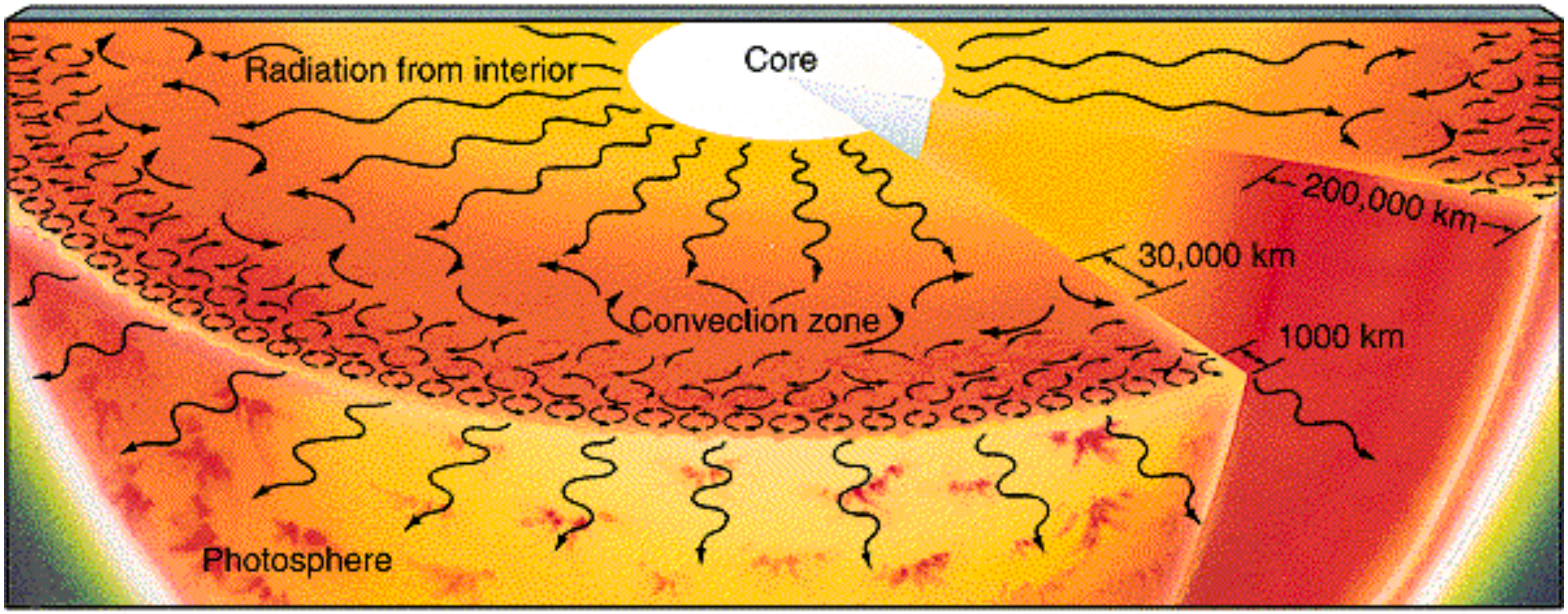


Zero-age main sequence stars
1.6
1.2
0.8
0.4
0
–0.4
$\log\left(\frac{M}{M_\odot}\right)$
Massive star
Sun
0
0.2
0.4
0.6
0.8
1
m/M
center
surface

Convection

Very low mass Main sequence	Fully convective
Low mass Main sequence	Radiative core Convective surface
High mass Main sequence	Convective core Radiative surface
White dwarfs	Conductive

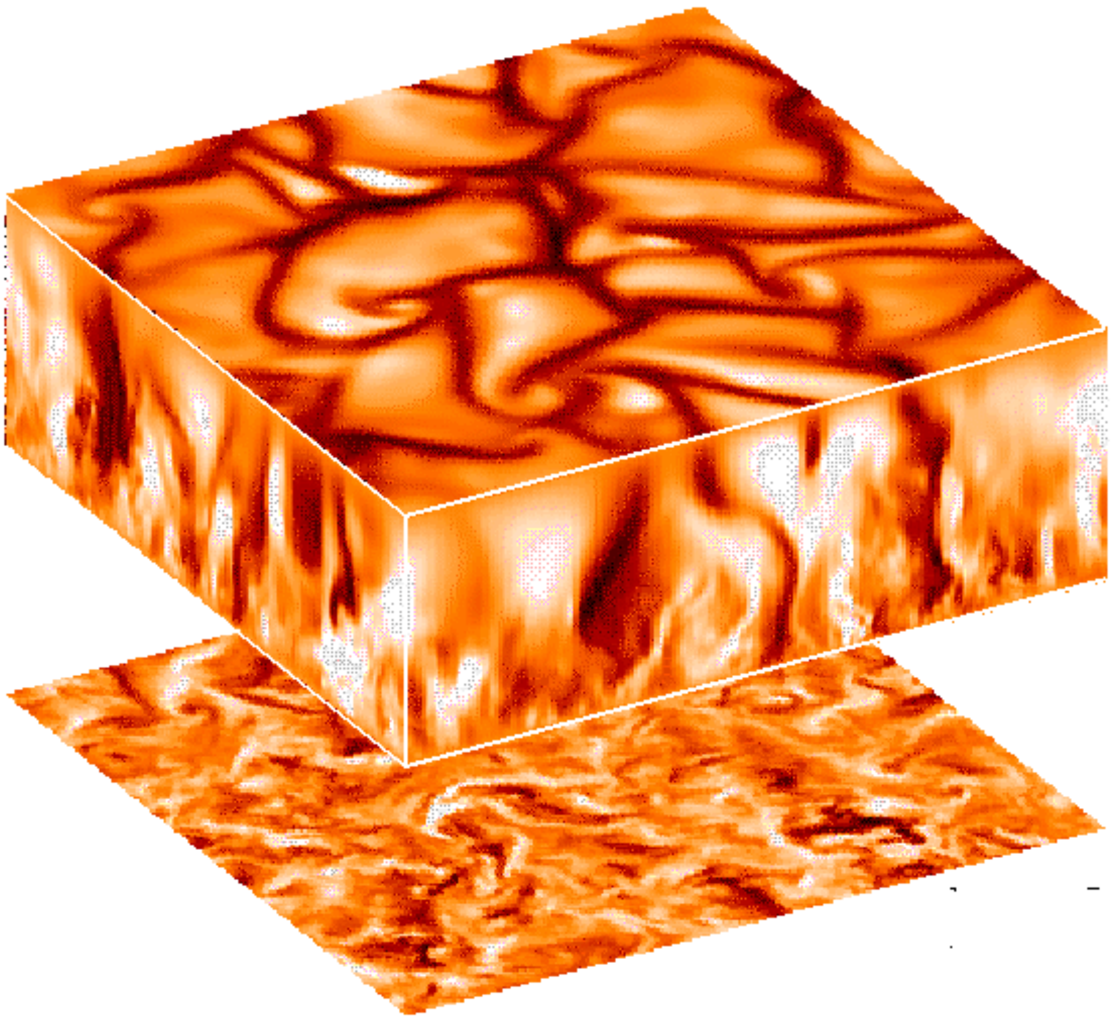
Convection



Core
Radiation from interior
200,000 km
30,000 km
Convection zone
1000 km
Photosphere

Convection

Convection is incredibly difficult to model accurately because fluid motions are very complicated. It is a 3D problem.



Convection

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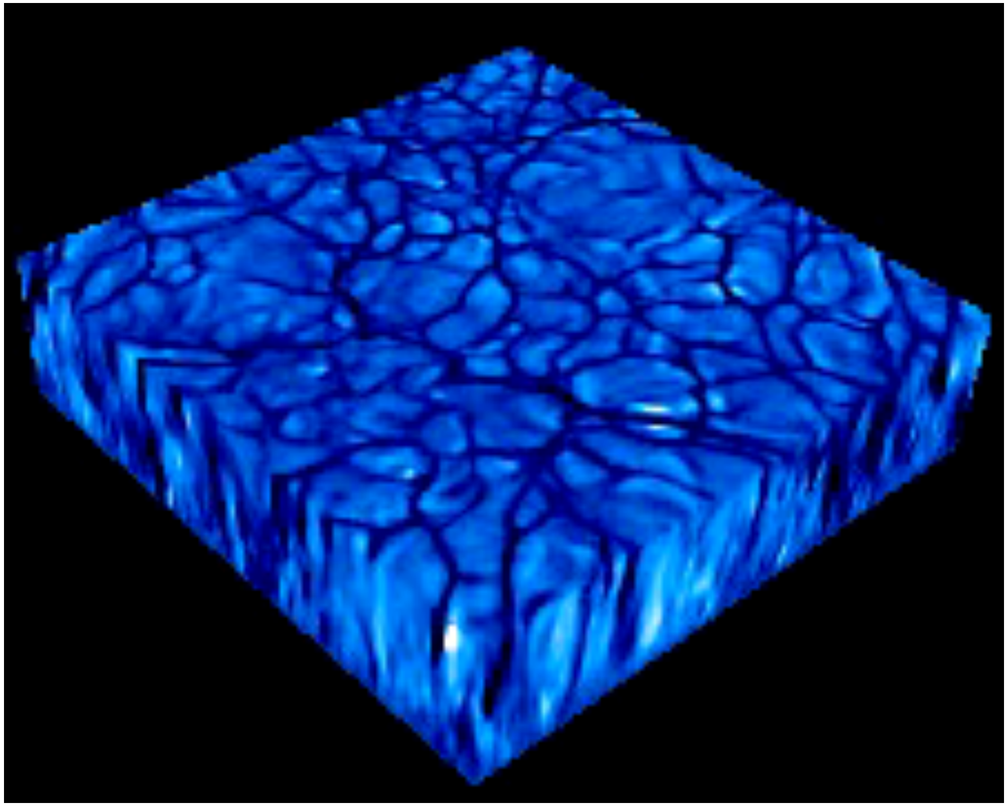
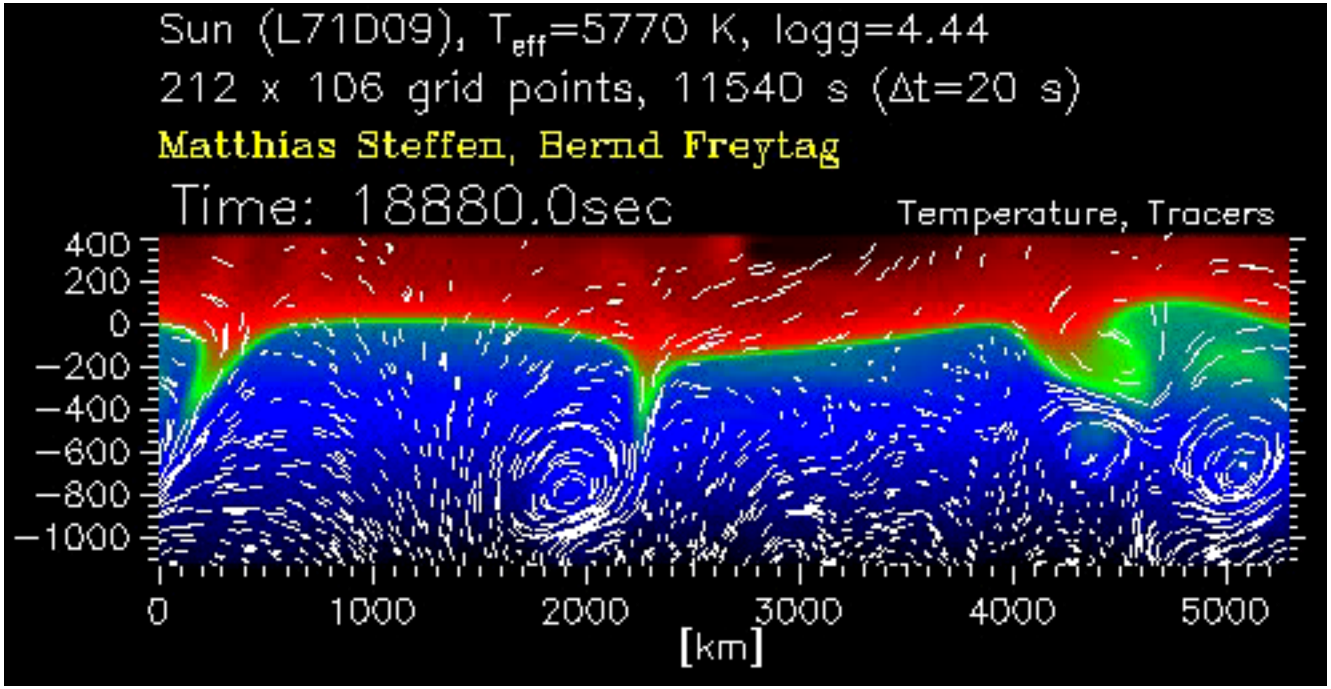


Convection

Convection is incredibly difficult to model accurately because fluid motions are very complicated. It is a 3D problem.



Convection: Energy Flux

- No reliable analytic way to compute

- Lab experiments with fluids are not relevant
    - incompressible
    - too viscous
    - boundaries

- Hydrodynamic simulations are challenging

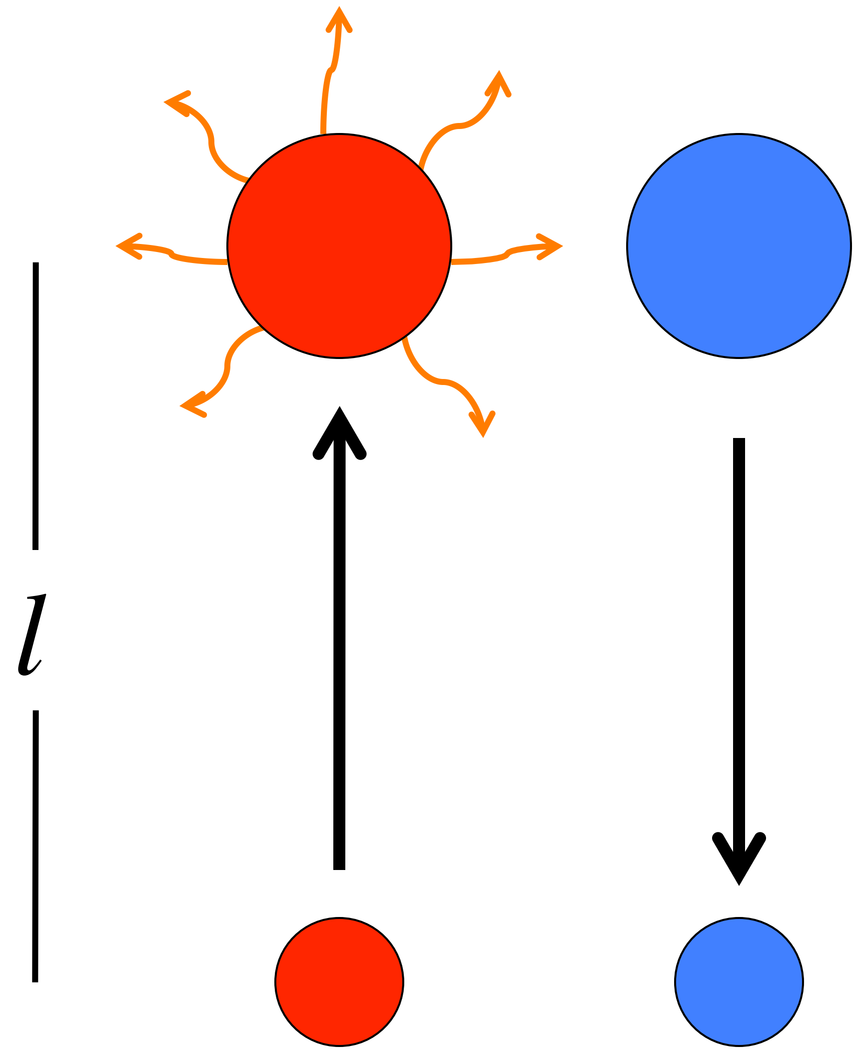
- Empirical simple model for spherically symmetric stellar models: Mixing Length Theory

Convection: Mixing Length Theory

<u>1 parameter</u>: length of mixing

Each mass element rises or falls a distance l adiabatically

After one mixing length, The element thermalizes with the local environment.



Convection: Mixing Length Theory

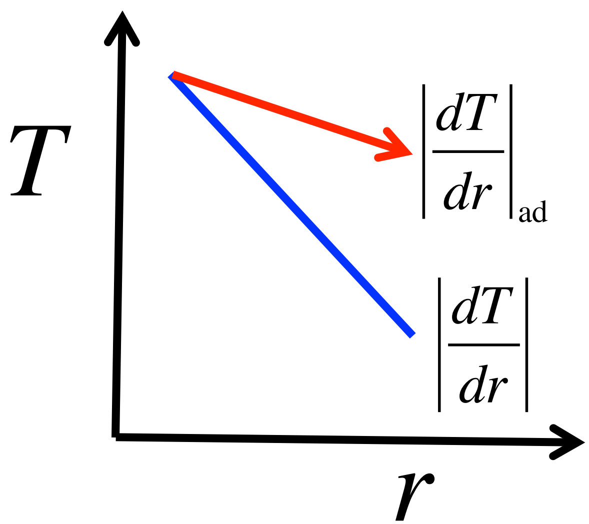
- l should scale with the pressure scale height

$$\lambda_P = P\left|\frac{dP}{dr}\right|^{-1}$$ (length over which pressure changes by 100%)

$$l = \alpha \cdot \lambda_P \qquad \alpha: \text{ free parameter} \qquad (\alpha = 1.6 \pm 0.1)$$

- After rising adiabatically by a distance l, the mass element will be hotter than its surrounding gas by

$$\Delta T = \left(\left|\frac{dT}{dr}\right| - \left|\frac{dT}{dr}\right|_{\mathrm{ad}}\right) \cdot l \equiv l \cdot \Delta\nabla T$$



Convection: Mixing Length Theory

- If the mass element thermalizes at constant pressure, the amount of heat per unit mass released is

$$\Delta Q = c_P \Delta T = c_P l \Delta \nabla T$$

- If the average velocity of the mass element is $\bar{v}$, the average excess heat flux is

$$F_{\text{conv}} = \rho \bar{v} \Delta Q = \rho \bar{v} c_P l \Delta \nabla T$$

$$\frac{\text{heat}}{\text{mass}} \times \frac{\text{mass}}{\text{volume}} \times \frac{\text{length}}{\text{time}} = \frac{\text{heat}}{\text{area} \cdot \text{time}}$$

- The mass element is accelerated due to buoyancy $\left(\rho < \rho_{\text{ambient}}\right)$

$$\Delta \rho = \left(\left| \frac{d\rho}{dr} \right| - \left| \frac{d\rho}{dr} \right|_{\text{ad}} \right) l \equiv l \Delta \nabla \rho \quad \rightarrow \overline{\Delta \rho} = \frac{1}{2} l \Delta \nabla \rho$$

Convection: Mixing Length Theory

- The average buoyant force per unit volume is

$$\bar{F} = g \cdot \overline{\Delta\rho} \quad = \frac{1}{2} g l \Delta\nabla\rho$$

where $g = \dfrac{Gm}{r^2}$

- The resulting acceleration is

$$\bar{a} = \frac{\bar{F}}{\rho} \quad = \frac{gl}{2\rho}\Delta\nabla\rho \quad = \frac{Gml}{2\rho r^2}\Delta\nabla\rho$$

- If the mass element starts at rest and has a constant acceleration

$$\left.\begin{aligned} v &= \bar{a} \cdot t \\ t &= \frac{l}{\bar{v}} = \frac{l}{\frac{1}{2}v} = \frac{2l}{v} \end{aligned}\right\} \rightarrow v = \left(2\bar{a}l\right)^{1/2} \quad \rightarrow \bar{v} = \frac{1}{2}\left(2\bar{a}l\right)^{1/2}$$

$$\rightarrow \bar{v} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\Delta\nabla\rho\right)^{1/2}$$

Convection: Mixing Length Theory

$$F_{\text{conv}} = \rho \bar{v} c_P l \Delta\nabla T$$

$$\bar{v} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\Delta\nabla\rho\right)^{1/2}$$

$$F_{\text{conv}} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\Delta\nabla\rho\right)^{1/2} \rho c_P l \Delta\nabla T$$

$$\Delta\nabla\rho \equiv \left(\left|\frac{d\rho}{dr}\right| - \left|\frac{d\rho}{dr}\right|_{\text{ad}}\right) \approx \frac{\rho}{T}\left(\left|\frac{dT}{dr}\right| - \left|\frac{dT}{dr}\right|_{\text{ad}}\right) \equiv \frac{\rho}{T}\Delta\nabla T$$

$$F_{\text{conv}} = \frac{l}{2}\left(\frac{Gm}{\rho r^2}\frac{\rho}{T}\Delta\nabla T\right)^{1/2} \rho c_P l \Delta\nabla T$$

$$F_{\text{conv}} = \frac{l^2}{2} c_P \rho \left(\frac{Gm}{Tr^2}\right)^{1/2} \left(\Delta\nabla T\right)^{3/2}$$

Convection: Mixing Length Theory

- How large an excess T gradient is needed to carry luminosity?

$$\left(\Delta\nabla T\right)^{3/2} = \frac{\dfrac{L_r}{4\pi r^2}}{\dfrac{l^2}{2} c_P \rho \left(\dfrac{Gm}{Tr^2}\right)^{1/2}} = \frac{L_r}{2\pi l^2 c_P \rho r \left(Gm/T\right)^{1/2}}$$

$$\text{e.g., at } \frac{m}{M} = 0.5,\ l = \lambda_P = 5\times 10^9\,\text{cm},\ \text{solar values for } T, \rho, r, L_r, c_P$$

$$\Delta\nabla T \sim 10^{-10}\,\text{Kcm}^{-1} \qquad \left|\frac{dT}{dr}\right| \approx \frac{T_c}{R} \sim 10^{-4}\,\text{Kcm}^{-1}$$

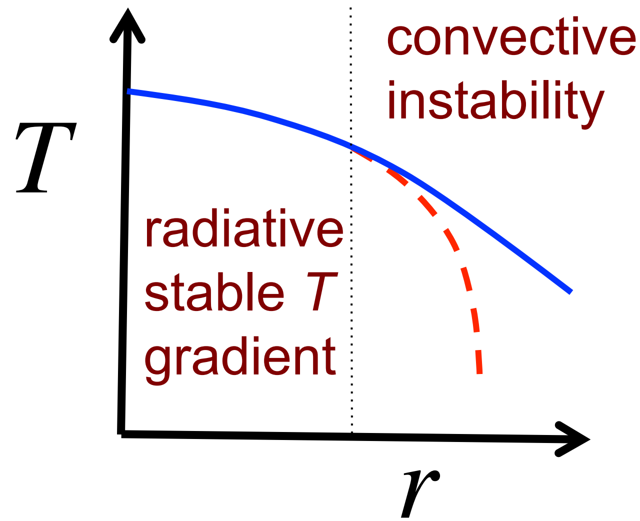
Requires an excess of only ~10^{-6} of the temperature gradient!

Convection: Mixing Length Theory

- When convection occurs in stellar interiors, the resulting temperature gradient equals the adiabatic gradient.

$$\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{\text{ad}} + \Delta\nabla T$$

$$\approx \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dT}$$



- If composition changes with radius, this may be different. e.g., if μ drops with r → heavier material is displaced into lighter material → greater stability.

Nuclear burning → stabilizing

Convection: Mixing Length Theory

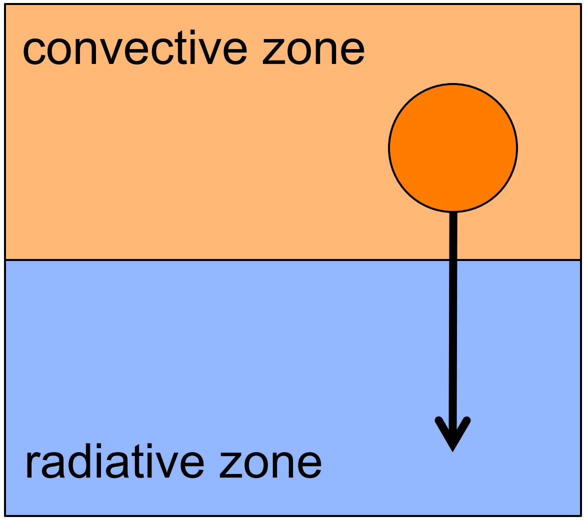
- When convection is efficient, the exact value of the mixing length l does not matter.

  Near surface, λ_P is small → larger $\Delta\nabla T$ is needed.

- Stellar radius *R*, surface *T*, etc. depend on α

  x2 change in α → few hundred K difference in T_{eff}.

Convection: Mixing Length Theory



Convective Overshooting

Do convective elements overshoot into stable regions due to inertia?

Naively: No because acceleration is small and braking is large in stable zone.

Simulations: Overshooting extends to significant fraction of λ_P

Alternative to Mixing Length model

"Full spectrum of turbulence"

Turbulent processes involve a range of length scales.

It is unclear if this works better than the one-parameter model.