Energy Generation: Gravitational Sources

In a static star, the energy source is thermonuclear

$$\frac{dL_m}{dm} = \varepsilon$$

In a contracting/expanding star, if the process is not adiabatic, there is an extra source/sink of energy so that

$$\frac{dL_m}{dm} \neq \varepsilon$$

Instead: $$\frac{dL_m}{dm} = \varepsilon + \varepsilon_{\mathrm{grav}}$$

In real stars, contraction/expansion is a local process. e.g., core contracts while envelope expands → $\varepsilon_{\mathrm{grav}}$ is a function of radius.

$$\varepsilon_{\mathrm{grav}} = \frac{d\Omega}{dt \cdot dm} = \frac{d}{dt}\left(\frac{Gm}{r}\right) = -\frac{Gm}{r^2}\dot{r}$$

Energy Generation: Neutrino Losses

L_m excludes energy flux in neutrinos.

Matter at normal densities and temperatures is completely transparent to neutrinos → energy loss

neutrino cross-section: $\sigma_\nu \sim 10^{-44} \varepsilon_\nu^2 \text{ cm} \ \left(\varepsilon_\nu : \text{ energy in MeV}\right)$

$$\lambda = \frac{1}{n\sigma_\nu} \qquad \lambda \sim 10^{20} \varepsilon_\nu^{-2} \rho^{-1} \text{cm}$$

It is only possible to get a short mean-free-path if the density is very high. This is only true in the cores of SN where neutrino pressure is important.

$$\boxed{\frac{dL_m}{dm} = \varepsilon + \varepsilon_{\text{grav}} - \varepsilon_\nu}$$

Nuclear Energy Generation

- Lighter nuclei with mass M_j fuse to form a heavier nucleus of mass M_y . The energy liberated is

$$E = \Delta M c^2 \quad = \left(\sum_j M_j - M_y \right) c^2$$

$$4 \times {}^1H \ \text{(mass 1.0079 amu)} \ \rightarrow \ {}^4He \ \text{(mass 4.0026 amu)}$$

$$\Delta M \sim 0.7\% \text{ or original masses, or 26.5MeV}$$

- The mass of a nucleus is not simply equal to the sum of its constituents (i.e., protons and neutrons). There is also binding energy that holds the constituents together.

Nuclear Energy Generation

- The binding energy of a nucleus is defined to be

$$E_B = \left(\text{mass of constituent nucleons} \;-\; \text{mass of bound nucleus}\right)c^2$$

$$\boxed{E_B = \left[\left(A - Z\right)m_n + Zm_p - M_{\text{nuc}}\right]c^2}$$

m_n : neutron mass

m_p : proton mass

- The binding energy per nucleon is

$$f = \frac{E_B}{A}$$

- Binding energy = energy required to separate the nucleus to infinity against binding forces.

Nuclear Energy Generation

- To get energy from fusing lighter nuclei into heavier nuclei, the total binding energy of the light nuclei must be lower than that of the heavier nucleus. $\left(M_{\text{nuc}} \text{for light} > M_{\text{nuc}} \text{for heavy}\right)$

$$\text{e.g., } 3\times {}^{4}He \rightarrow {}^{12}C \qquad 3\times {}^{4}He: \quad 3E_{B,He} = 3\times\left(2n+2p-M_{He}\right)c^2$$

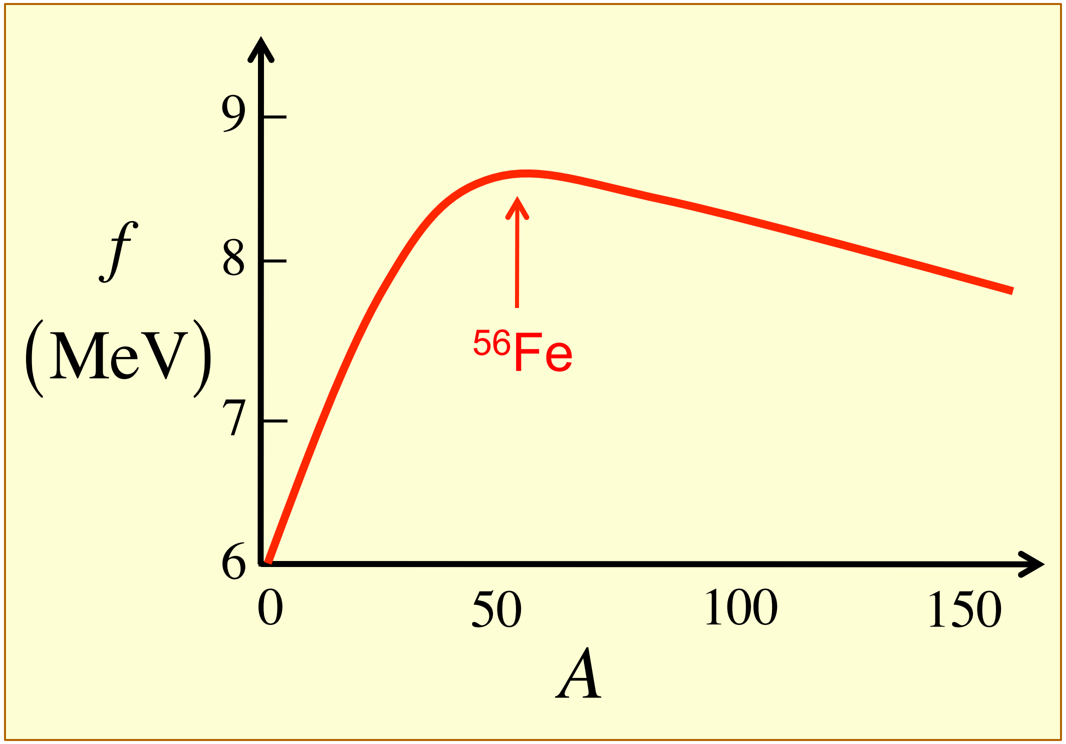
$$ {}^{12}C: \quad E_{B,C} = \left(6n+6p-M_C\right)c^2$$

$$3M_{He} > M_C \rightarrow 3E_{B,He} < E_{B,C} \quad \rightarrow \frac{E_{B,He}}{4} < \frac{E_{B,C}}{12}$$

- A nuclear reaction can release energy if $\dfrac{E_B}{A}$ for light element is less than that for heavy element.

Nuclear Energy Generation

- The most tightly bound nucleus is ^{56}Fe. Fusion of pure ^{1}H to ^{56}Fe yields ~8.5MeV per nucleon, the largest part of which (6.6MeV) is already obtained in fusion to ^{4}He.



- Fusion of elements with A<56 yields energy.
- Fission of elements with A>56 yields energy.
- Elements near ^{56}Fe are the most tightly bound and are thus not much use for energy production.
- If a star ends up with ^{56}Fe, energy production is over.

Nuclear Energy Generation: Coulomb Barrier

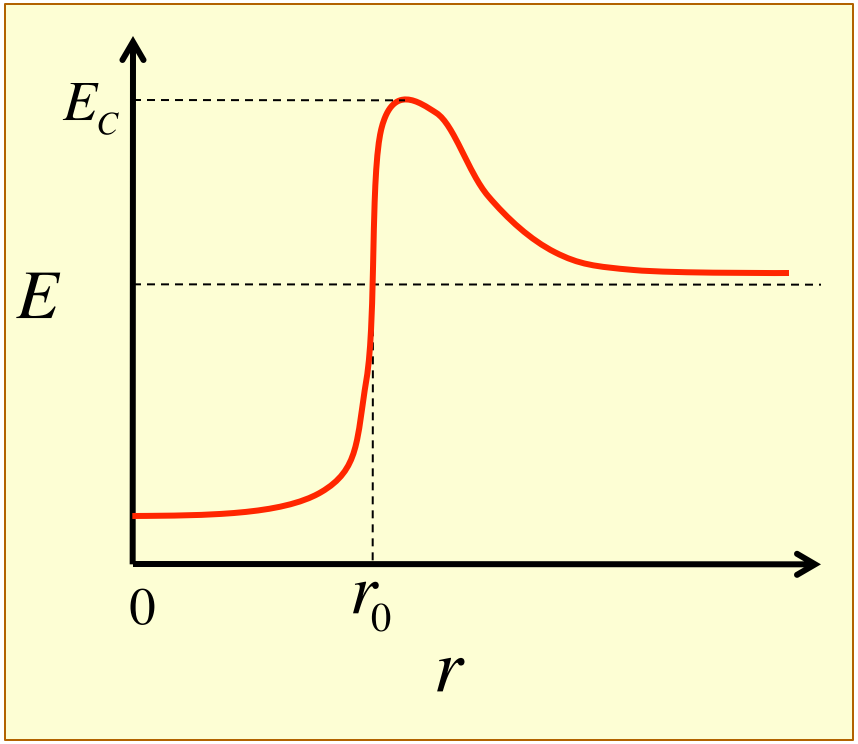
- To fuse, nuclei must surmount a Coulomb barrier.

- Nuclear forces dominate within a radius

$$r_0 = 1.44 \times 10^{-13} A^{1/3} \mathrm{cm}$$

- For nuclei of charge Z_1 and Z_2 height of Coulomb barrier is

$$E_C = \frac{Z_1 Z_2 e^2}{r_0} \sim Z_1 Z_2 \ \mathrm{MeV}$$



- At T=10^7 K, the thermal energy kT is ~10^3eV → classically there are ZERO particles in a thermal distribution with sufficient energy to fuse at this T.

Nuclear Energy Generation: Tunneling

- Quantum tunneling is required for fusion to take place. Tunneling probability is

$$P = P_0 E^{-1/2} e^{-2\pi\eta} \qquad \eta = \left(\frac{m}{2}\right)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar E^{1/2}}$$

P_0 : depends on nuclei

m : reduced mass

P increases rapidly with energy

P decreases with Z_1, Z_2 → lightest elements can fuse at lowest temperatures.

- Higher energies and temperatures are needed to fuse heavier nuclei → well separated phases in which different elements burn during stellar evolution.

$$T\downarrow \quad \begin{array}{l} H \rightarrow He \\ He \rightarrow C,O \\ C,O \rightarrow Na,Ne,Si,P \end{array}$$

Nuclear Energy Generation: Resonances

- Most thermonuclear reactions in stars proceed through an intermediate state called the "compound nucleus"

$$\alpha + X \rightarrow Z^* \rightarrow Y + \beta$$

α : projectile (proton or α particle)

X : target nucleus

Z^* : compound nucleus in excited state

- Reactions happen in several steps:
  - Tunneling through the Coulomb barrier
  - Formation of a compound excited nucleus Z^*, whose energy depends on the reaction and kinetic energy of reacting particles.
  - Decay of the Z^* via emission of photons, neutrons, protons, alpha particles, electrons, etc.

Nuclear Energy Generation: Resonances

e.g.,

$$\begin{aligned} p + {}^{11}B \rightarrow {}^{12}C^{*} &\rightarrow {}^{12}C^{**} + \gamma \\ &\rightarrow {}^{11}B + p \\ &\rightarrow {}^{11}C + n \\ &\rightarrow {}^{12}N + e^{-} + \overline{\nu}_e \\ &\rightarrow {}^{8}Be + {}^{4}He \end{aligned}$$

- The probability of a given output depends on the decay lifetime over the sum of all possible lifetimes.

If τ_i is the lifetime of a particular output, then the probability of that output is:

$$P_i = \frac{1/\tau_i}{\sum_i 1/\tau_i} = \frac{\tau}{\tau_i} \quad \text{where} \quad \tau = \left(\sum_i 1/\tau_i \right)^{-1} : \text{total mean life of } {}^{12}\mathrm{C}^{*}$$

Nuclear Energy Generation: Resonances

- Through the uncertainty principle, the energy width is $\Gamma_i \tau_i = \hbar$

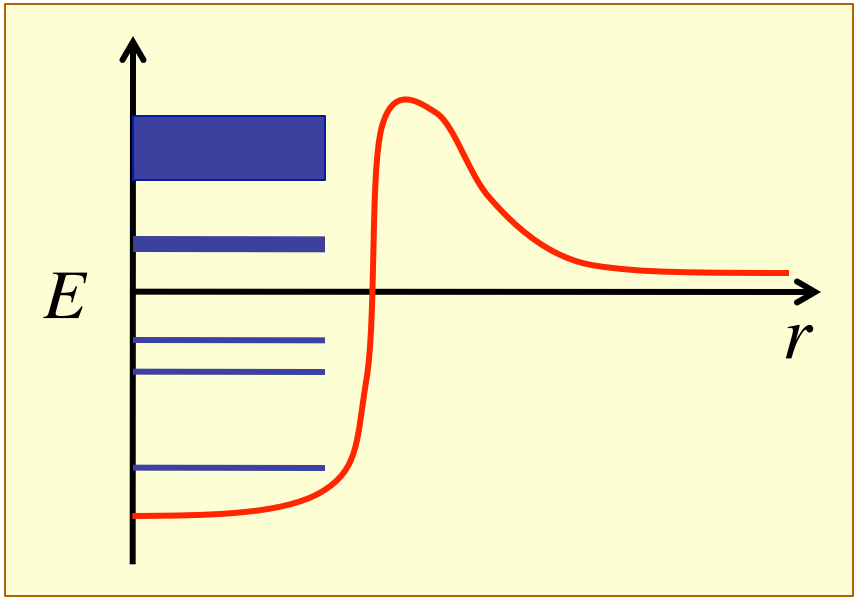
$$P_i = \frac{\Gamma_i}{\sum_j \Gamma_j} = \frac{\Gamma_i}{\Gamma}$$

where Γ : total energy width of $^{12}\mathrm{C}^*$

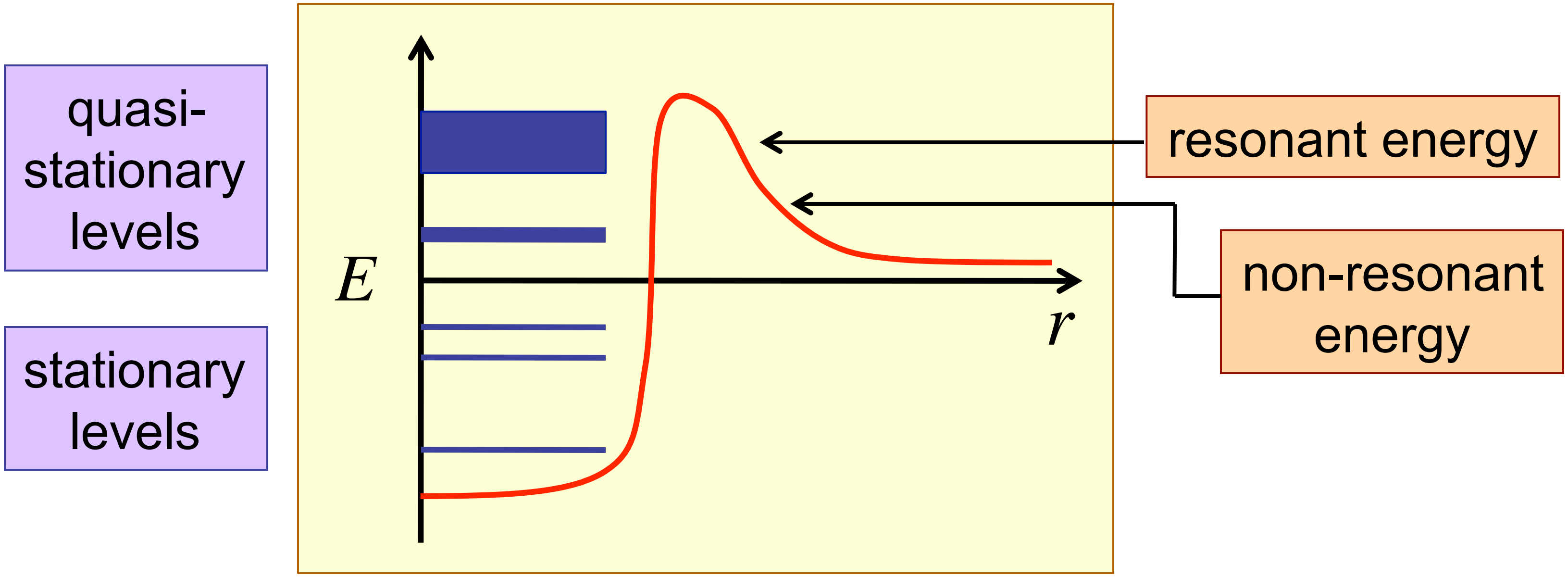
- The compound nucleus has

Stationary energy levels corresponding to excited atomic states that can decay via photon emission.

Quasi-stationary levels, which can decay via particles tunneling back through the Coulomb barrier.



Nuclear Energy Generation: Resonances



- The lifetimes of quasi-stationary states are shorter and they have broader energy ranges.

- At high energy, the width of states increases so that energy levels overlap → continuum of excited levels.

Nuclear Energy Generation: Resonances

- The cross-section for some astrophysically interesting reactions depends critically upon the energy level structure of the compound nucleus. At resonant energies, σ can be boosted by orders of magnitude.

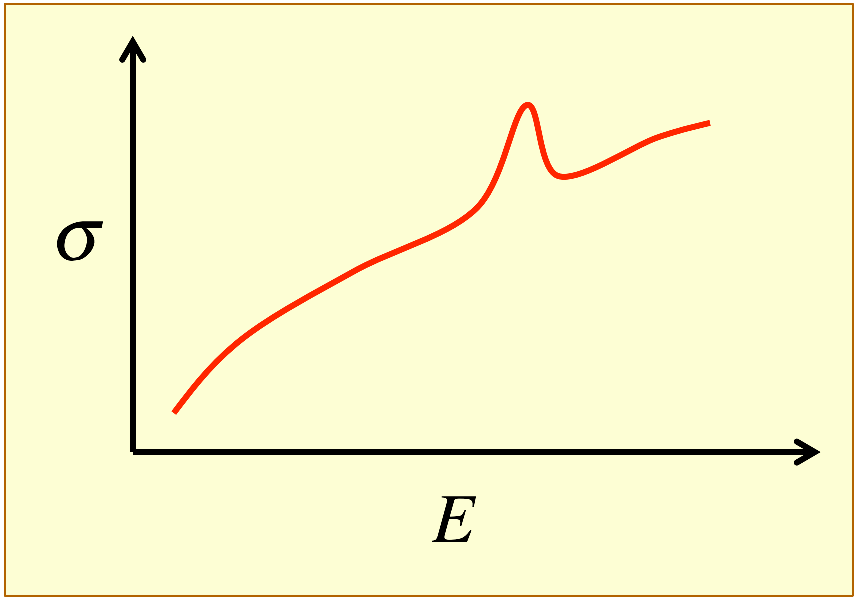
- The maximum cross-section is the geometric cross-section

$$\pi\lambda^2 \propto E^{-1}$$

- Also, add dominant exponential tunneling factor

$$\boxed{\sigma(E) = S(E)E^{-1}e^{-2\pi\eta}}$$

S : astrophysical cross-section: varies very slowly with E



Nuclear Energy Generation: Reaction Rates

- The energy generation rate depends on the energy released per reaction times the reaction rate.

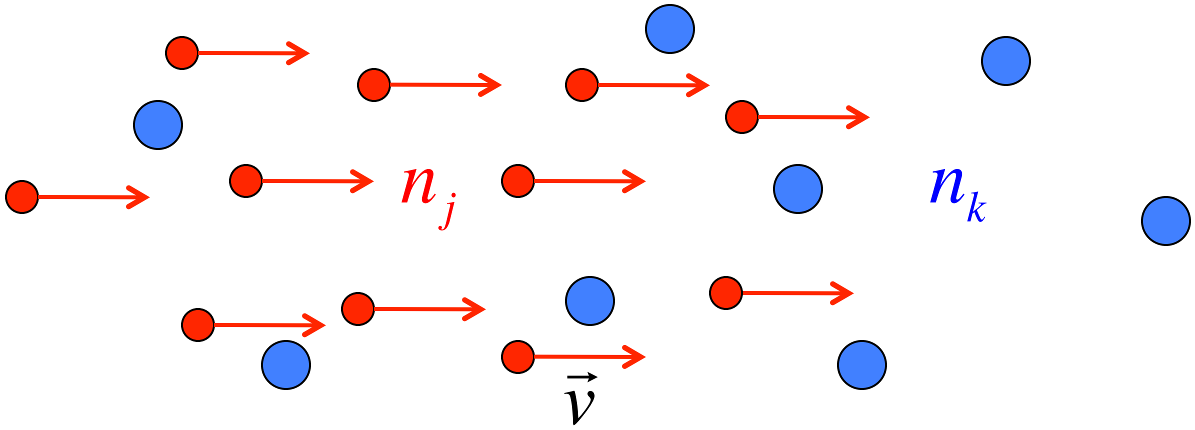
- Which energies of a particle contribute most to the total reaction rate?

  High *E* gives higher probability for fusion per reaction, but fewer potential reactions

- Why do nuclear reactions have a high *T* dependence?

Nuclear Energy Generation: Reaction Rates

- Consider particles of type *j* moving with velocity *v* relative to particles of type *k*. The number densities are n_j and n_k.



- The number of reactions per unit time per volume is:

$$\tilde{r}_{jk} = n_j \cdot n_k \cdot \sigma \cdot v$$

- To avoid double counting:

$$\boxed{\tilde{r}_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \sigma v}$$

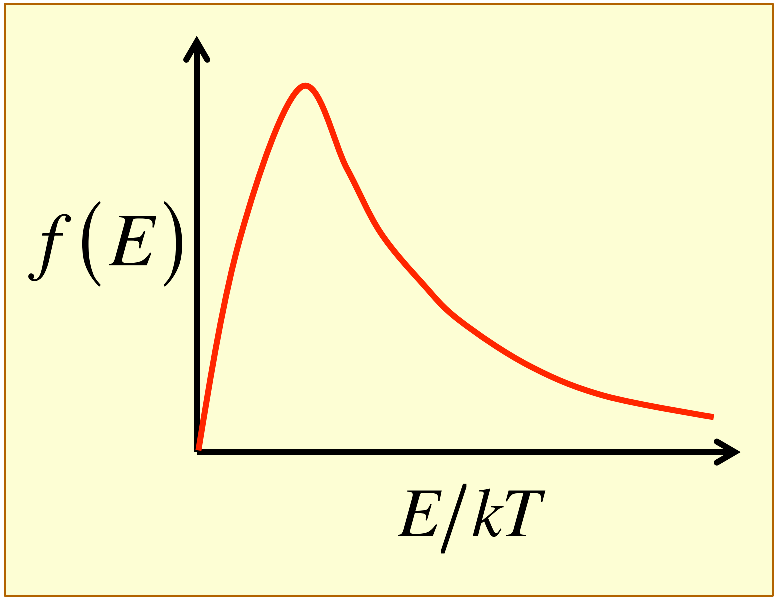
Nuclear Energy Generation: Reaction Rates

- Assume that both species have Maxwell-Boltzmann velocity distributions → the relative velocity is also Maxwellian.

$$E = \frac{1}{2} m v^2, \quad \text{where } m \text{ is the reduced mass:} \quad m = \frac{m_j m_k}{m_j + m_k}$$

- The fraction of all pairs with energy between *E* and *E*+*dE*:

$$\boxed{f(E)\,dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT}\,dE}$$



Nuclear Energy Generation: Reaction Rates

- The total reaction rate (per volume, per time) is the mono-energetic rate integrated over all energies and weighted by *f(E)*

$$r_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \langle \sigma v \rangle$$

where

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E) \cdot v \cdot f(E) dE$$

- Replace number densities with mass fractions: $X_i \rho = n_i m_i$

- If each reaction releases an amount of energy Q, then:

$$\boxed{\varepsilon_{jk} = \frac{1}{1+\delta_{jk}} \frac{Q}{m_j m_k} \rho X_j X_k \langle \sigma v \rangle}$$

$$\left(\frac{\text{energy}}{\text{time} \cdot \text{mass}} \right)$$

Nuclear Energy Generation: Reaction Rates

- All the temperature dependence is contained in $\langle \sigma v \rangle$

$$\sigma(E) = S(E)E^{-1}e^{-2\pi\eta}$$

$$v = \left(\frac{2E}{m}\right)^{1/2}$$

$$f(E)dE = \frac{2}{\sqrt{\pi}}\frac{E^{1/2}}{(kT)^{3/2}}e^{-E/kT}dE$$

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E)\cdot v \cdot f(E)dE$$

$$= \int_0^\infty S(E)\cancel{E}^{-1}e^{-2\pi\eta}\left(\frac{2}{m}\right)^{1/2}\cancel{E}^{1/2}\frac{2}{\sqrt{\pi}}\frac{\cancel{E}^{1/2}}{(kT)^{3/2}}e^{-E/kT}\,dE$$

$$= \frac{2^{3/2}}{(\pi m)^{1/2}}\frac{1}{(kT)^{3/2}}\int_0^\infty S(E)e^{-2\pi\eta}e^{-E/kT}\,dE$$

Nuclear Energy Generation: Reaction Rates

$$\langle \sigma v \rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E)\, e^{-2\pi\eta} e^{-E/kT}\, dE$$

$$\bar{\eta} = 2\pi\eta E^{1/2} \quad = \pi (2m)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar}$$

$$\eta = \left(\frac{m}{2}\right)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar E^{1/2}}$$

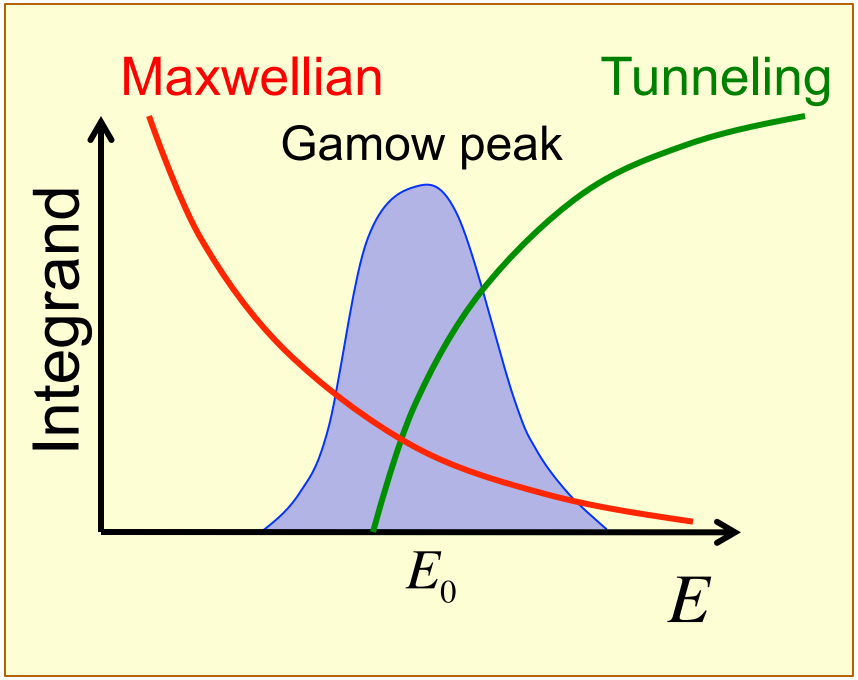
$$\langle \sigma v \rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E)\, e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)}\, dE$$

- Ignore *S(E)*. As *E* goes up, the energy distribution term (Maxwellian) drops, and the tunneling term increases.

Nuclear Energy Generation: Reaction Rates

$$\text{Integrand} = S(E)\, e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)}$$

- Integrand peaks at E_0, where $d(\text{Integrand})/dE = 0$
- Since only a narrow range of E contributes to the integral, it is ok to assume $S(E) \approx S_0 = \text{const}$ (for non-resonant reactions)



$$\frac{d(\text{Integrand})}{dE} = e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)} \left(-\frac{1}{kT} + \frac{\bar{\eta}}{E}\frac{1}{2}E^{-1/2} \right)$$

- This is zero when: $\dfrac{\bar{\eta}}{E}\dfrac{1}{2}E^{-1/2} = \dfrac{1}{kT} \quad \rightarrow E_0 = \left(\dfrac{\bar{\eta} kT}{2} \right)^{2/3}$

Nuclear Energy Generation: Reaction Rates

- Evaluate the integral $\int_0^\infty e^{\left(-E/kT-\bar{\eta}/E^{1/2}\right)}\,dE = \int_0^\infty e^{f(E)}\,dE$

- Expand *f(E)* about the maximum at *E*=*E*$_0$ using a series expansion truncated at the quadratic term

$$\bar{\eta} = \frac{2E_0^{3/2}}{kT}$$

$$f(E) = f(E_0) + \cancel{f'(E_0)(E-E_0)} + \frac{1}{2}f''(E_0)(E-E_0)^2 + \ldots$$

$$f(E_0) = -\frac{E_0}{kT} - \frac{\bar{\eta}}{E_0^{1/2}} \quad = -\frac{E_0}{kT} - \frac{2E_0}{kT} \quad = -3\frac{E_0}{kT}$$

$$f'(E_0) = -\frac{1}{kT} + \frac{\bar{\eta}}{2E_0^{3/2}} \quad = 0$$

$$f''(E_0) = -\frac{\bar{\eta}}{2E_0^3}\frac{3}{2}E_0^{1/2} \quad = -\frac{3\bar{\eta}}{4E_0^{5/2}} \quad = -\frac{3}{2E_0kT}$$

Nuclear Energy Generation: Reaction Rates

$$f(E) = f(E_0) + \frac{1}{2} f''(E_0)(E - E_0)^2 + \ldots$$

$$= -3\frac{E_0}{kT} + \frac{1}{2}\left(-\frac{3}{2E_0 kT}\right)(E - E_0)^2 + \ldots$$

$$= -3\frac{E_0}{kT} - \frac{3}{4E_0 kT} E_0^2 \left(\frac{E}{E_0} - 1\right)^2 + \ldots$$

$$= -3\frac{E_0}{kT} - \frac{3E_0}{4kT}\left(\frac{E}{E_0} - 1\right)^2 + \ldots$$

$$\tau = 3\frac{E_0}{kT}$$

$$= -\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2 + \ldots$$

$$\int_0^\infty e^{f(E)}\, dE = \int_0^\infty \exp\left[-\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2\right] dE$$

Nuclear Energy Generation: Reaction Rates

$$\int_0^\infty e^{f(E)}\,dE = \int_0^\infty \exp\left[-\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2\right] dE$$

$$= e^{-\tau}\int_0^\infty \exp\left[-\frac{\tau}{4}\left(\frac{E}{E_0} - 1\right)^2\right] dE$$

$$= e^{-\tau}\int_0^\infty e^{-\frac{\tau}{4}x^2} E_0\,dx \qquad = E_0 e^{-\tau}\int_0^\infty e^{-\frac{\tau}{4}x^2}\,dx \qquad = E_0 e^{-\tau}\left(\frac{\pi}{\tau}\right)^{1/2}$$

$$= \frac{\tau kT}{3} e^{-\tau}\left(\frac{\pi}{\tau}\right)^{1/2} \qquad = \frac{\pi^{1/2}}{3} kT\tau^{1/2} e^{-\tau}$$

$$x = \left(\frac{E}{E_0} - 1\right)$$

$$dx = \frac{dE}{E_0}$$

$$\boxed{\int_0^\infty e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)}\,dE = \frac{\pi^{1/2}}{3} kT\tau^{1/2} e^{-\tau}}$$

Nuclear Energy Generation: Reaction Rates

- Plug this into the expression for the cross-section

$$\langle\sigma v\rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{S_0}{(kT)^{3/2}} \int_0^\infty e^{\left(-E/kT - \bar{\eta}/E^{1/2}\right)} dE$$

$$\langle\sigma v\rangle = \frac{2^{3/2}}{(\pi m)^{1/2}} \frac{1}{(kT)^{3/2}} \frac{\pi^{1/2}}{3} S_0 kT \tau^{1/2} e^{-\tau}$$

$$\boxed{\langle\sigma v\rangle = \frac{2^{2/3}}{3(m)^{1/2}} \frac{1}{(kT)^{1/2}} S_0 \tau^{1/2} e^{-\tau}}$$

$$\tau \sim \frac{E_0}{kT} \sim \frac{(kT)^{2/3}}{kT} = (kT)^{-1/3}$$

$$\boxed{\langle\sigma v\rangle \propto S_0 \tau^2 e^{-\tau}}$$

$$\rightarrow kT \sim \tau^{-3}$$

Nuclear Energy Generation: Reaction Rates

- Peak energy:

$$E_0 = \left(\frac{\bar{\eta} kT}{2} \right)^{2/3}$$

$$\bar{\eta} = \pi (2m)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar}$$

$$\frac{E_0}{kT} = \left(\frac{\bar{\eta} kT}{2} \right)^{2/3} \frac{1}{kT} = \left(\frac{\pi (2m)^{1/2} Z_1 Z_2 e^2}{2\hbar} \right)^{2/3} (kT)^{-1/3}$$

$$m = \frac{m_j m_k}{m_j + m_k} = \frac{A_j A_k}{A_j + A_k} m_p$$

$$\frac{E_0}{kT} = \left(\frac{\pi^2 e^4 m_p}{2\hbar^2} \right)^{1/3} \left(\frac{A_j A_k}{A_j + A_k} \right)^{1/3} (Z_1 Z_2)^{2/3} (kT)^{-1/3}$$

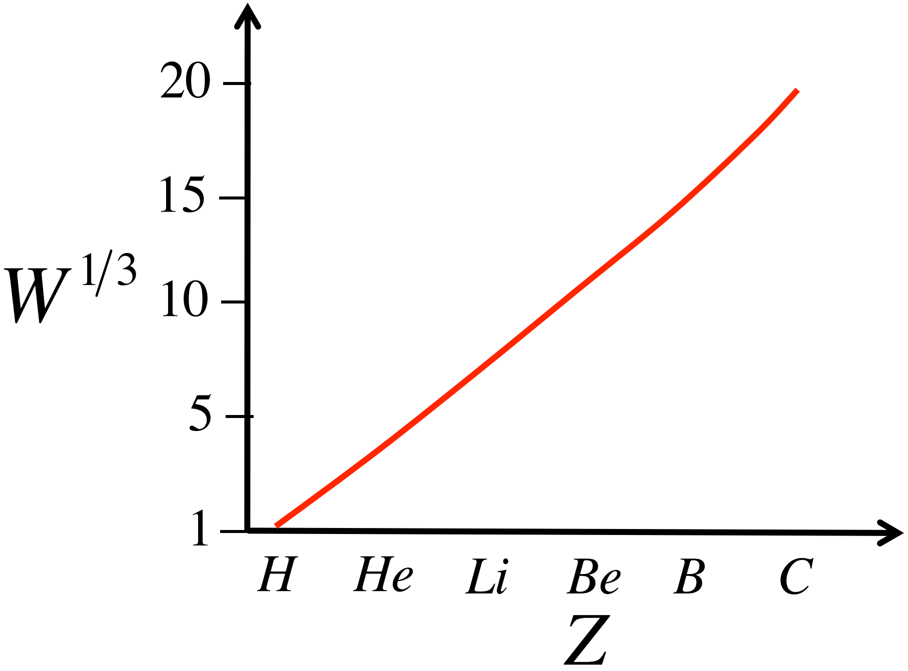
Nuclear Energy Generation: Reaction Rates

- Peak energy: $$\frac{E_0}{kT} = \left(\frac{\pi^2 e^4 m_p}{2\hbar^2}\right)^{1/3} \left(\frac{A_j A_k}{A_j + A_k}\right)^{1/3} \left(Z_1 Z_2\right)^{2/3} \left(kT\right)^{-1/3}$$

$$\boxed{\frac{E_0}{kT} \approx 6.6 W^{1/3} \left(\frac{T}{10^7\,\mathrm{K}}\right)^{-1/3}}$$

$$\boxed{W = Z_1^2 Z_2^2 \frac{A_j A_k}{A_j + A_k}}$$

The peak energy increases with W → separated phases of nuclear burning



Nuclear Energy Generation: Reaction Rates

$$\varepsilon_{jk} \sim \langle \sigma v \rangle \sim \tau^2 e^{-\tau}$$

$$\tau \sim T^{-1/3}$$

- Temperature dependence of energy generation rate

$$\boxed{\varepsilon_{jk} \sim T^{\nu}} \rightarrow \nu = \frac{\partial \ln \varepsilon_{jk}}{\partial \ln T} = \frac{\partial \ln \varepsilon_{jk}}{\partial \ln \tau} \frac{\partial \ln \tau}{\partial \ln T}$$

$$= \frac{\partial \left(2 \ln \tau - \tau \right)}{\partial \ln \tau} \frac{\partial \left(- \ln T / 3 \right)}{\partial \ln T} = \left(2 - \frac{\partial \tau}{\partial \ln \tau} \right) \left(-\frac{1}{3} \right)$$

$$= \left(2 - \tau \right) \left(-\frac{1}{3} \right) \rightarrow \boxed{\nu = \frac{\tau}{3} - \frac{2}{3}} \boxed{= 6.6 W^{1/3} \left(\frac{T}{10^7 \mathrm{K}} \right)^{-1/3} - \frac{2}{3}}$$

$\nu \approx 4$ for lightest elements, rising to $\nu \approx 40$ for heavier nuclei.

Strong *T* dependence → stars must be stable to *T* fluctuations.

Nuclear Energy Generation: Reaction Rates

- We can use reaction rates to compute the time derivatives of mass fractions.

Number of reactions per volume, per time: $$r_{jk} = \frac{1}{1+\delta_{jk}} X_j X_k \frac{\rho^2}{m_j m_k} \langle \sigma v \rangle$$

- Consider the reaction $A + B \rightarrow C$

The abundances of *A*, *B*, and *C* depend on $r_{A,B}$

- Each reaction yields one *C*, so the rate of change of the number density of *C* = $r_{A,B}$

$$\frac{dn_C}{dt} = r_{A,B} \quad \rightarrow \frac{dX_C}{dt}\frac{\rho}{m_C} = r_{A,B} \quad \rightarrow \frac{dX_C}{dt} = A_C \frac{m_p}{\rho} r_{A,B}$$

$$\rightarrow \frac{dX_C}{dt} = A_C \frac{m_p}{\rho} \frac{1}{1+\delta_{AB}} X_A X_B \frac{\rho^2}{m_A m_B} \langle \sigma v \rangle$$

Nuclear Energy Generation: Reaction Rates

$$\frac{dX_C}{dt} = A_C \frac{X_A X_B}{1+\delta_{AB}} \underbrace{\frac{\rho}{A_A A_B m_p}\langle\sigma v\rangle}_{r'_{A,B}} = A_C \frac{X_A X_B}{1+\delta_{AB}} r'_{A,B}$$

$$\frac{d\,^{16}O}{dt} = 16\left(^4He\,^{12}C \cdot r'_{4,12}\right)$$

$$\frac{d\,^{12}C}{dt} = 12\left(^4He\,^8Be \cdot r'_{4,8} - {}^4He\,^{12}C \cdot r'_{4,12}\right)$$

$$\frac{d\,^8Be}{dt} = 8\left(\frac{^4He\,^4He}{2} \cdot r'_{4,4} - {}^4He\,^8Be \cdot r'_{4,8}\right)$$

$$\frac{d\,^4He}{dt} = 4\left(-2\frac{^4He\,^4He}{2} \cdot r'_{4,4} - {}^4He\,^8Be \cdot r'_{4,8} - {}^4He\,^{12}C \cdot r'_{4,12}\right)$$

e.g.,

$$^4He + {}^4He \rightarrow {}^8Be$$

$$^8Be + {}^4He \rightarrow {}^{12}C$$

$$^{12}C + {}^4He \rightarrow {}^{16}O$$

Nuclear Energy Generation: Reaction Rates

- In “equilibrium”, the mass fractions of ^{8}Be and ^{12}C are conserved.

$$^{4}He + {}^{4}He \rightarrow {}^{8}Be$$

$$^{8}Be + {}^{4}He \rightarrow {}^{12}C$$

$$^{12}C + {}^{4}He \rightarrow {}^{16}O$$

$$\frac{d\,{}^{8}Be}{dt} = \frac{d\,{}^{12}C}{dt} = 0$$

$$\frac{{}^{4}He\,{}^{4}He}{2} \cdot r'_{4,4} = {}^{4}He\,{}^{8}Be \cdot r'_{4,8} \quad \rightarrow {}^{8}Be = \frac{1}{2}\,{}^{4}He \frac{r'_{4,4}}{r'_{4,8}}$$

$$^{4}He\,{}^{8}Be \cdot r'_{4,8} = {}^{4}He\,{}^{12}C \cdot r'_{4,12} \quad \rightarrow {}^{12}C = {}^{8}Be \frac{r'_{4,8}}{r'_{4,12}} \quad = \frac{1}{2}\,{}^{4}He \frac{r'_{4,4}}{r'_{4,12}}$$

The proton-proton chain

- Nuclear reactions usually involve several steps.
- The rate is controlled by the rate of the slowest reaction in the chain.
- The total energy release is the sum of energies of the individual steps.

The first step in the p-p chain is: $^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_e$

This reaction is non-resonant.

This reaction is slow because it involves a weak decay

While two protons are flashing past each other, one of them undergoes a weak decay into a neutron at that exact instant.

To make ^{4}He, we need 4 protons, two of which must be converted Into neutrons via either positron decays or e^{-} capture.

$$p \rightarrow n + e^{+} + \nu_e \qquad p + e^{-} \rightarrow n + \nu_e$$

The proton-proton chain

The reaction rate is:

$$r_{pp} = 1.15 \times 10^9 T_9^{-2/3} X^2 \rho^2 \exp\left(-3.38 / T_9^{1/3}\right) cm^{-3} s^{-1}$$

The temperature sensitivity is:

$$\nu_{pp} = \frac{11.3}{T_6^{1/3}} - \frac{2}{3}$$

In sun: $T_6 \approx 15 \rightarrow \nu_{pp} \approx 4$

The lifetime of a proton against destruction is:

$$\tau_p = -\frac{n_p}{dn_p / dt} = \frac{n_p}{2r_{pp}} \approx 6 \times 10^9 \, yr$$

For center of Sun

The proton-proton chain

pp1

$$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_e$$

$$^{2}H + {}^{1}H \rightarrow {}^{3}He + \gamma$$

$$^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$$

pp2

First two reactions of pp1

$$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$$

$$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_e$$

$$^{7}Li + {}^{1}H \rightarrow {}^{4}He + {}^{4}He$$

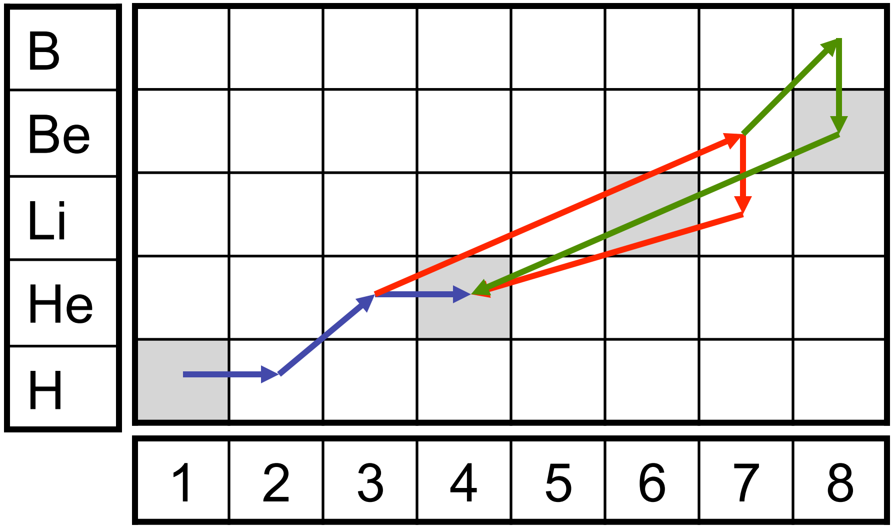
pp3

First three reactions of pp2

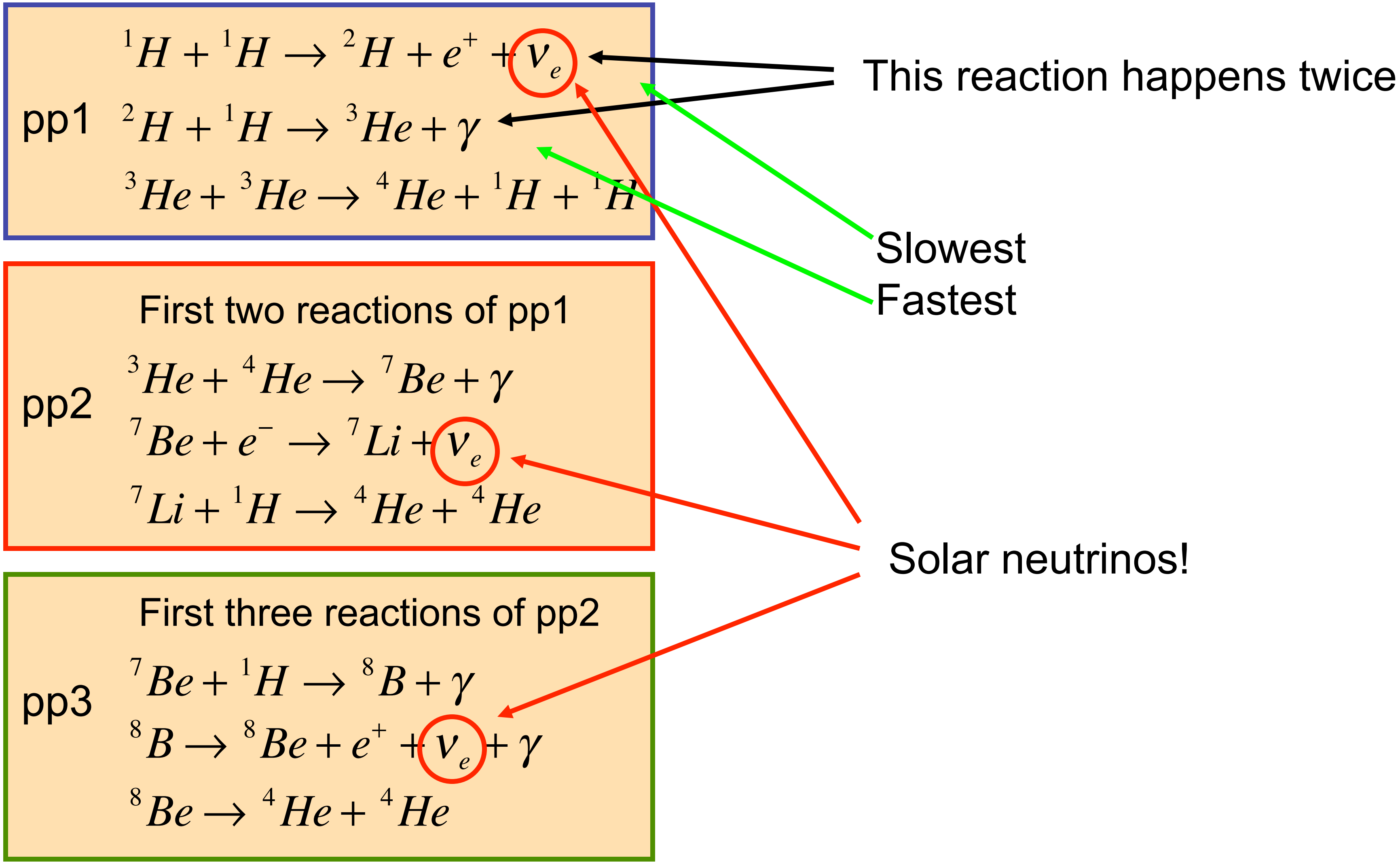
$$^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$$

$$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_e + \gamma$$

$$^{8}Be \rightarrow {}^{4}He + {}^{4}He$$



The proton-proton chain



pp1

$$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_{e}$$

$$^{2}H + {}^{1}H \rightarrow {}^{3}He + \gamma$$

$$^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$$

pp2

First two reactions of pp1

$$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$$

$$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_{e}$$

$$^{7}Li + {}^{1}H \rightarrow {}^{4}He + {}^{4}He$$

pp3

First three reactions of pp2

$$^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$$

$$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_{e} + \gamma$$

$$^{8}Be \rightarrow {}^{4}He + {}^{4}He$$

This reaction happens twice

Slowest

Fastest

Solar neutrinos!

The proton-proton chain

pp1

$$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_e$$

$$^{2}H + {}^{1}H \rightarrow {}^{3}He + \gamma$$

$$^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$$

pp2

First two reactions of pp1

$$^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$$

$$^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_e$$

$$^{7}Li + {}^{1}H \rightarrow {}^{4}He + {}^{4}He$$

pp3

First three reactions of pp2

$$^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$$

$$^{8}B \rightarrow {}^{8}Be + e^{+} + \nu_e + \gamma$$

$$^{8}Be \rightarrow {}^{4}He + {}^{4}He$$

pp1 is the most direct route but it involves the collision of two short lived ^{3}He nuclei.

Relative importance of chains thus depends on ^{3}He abundance.

As T increases, equilibrium abundance of ^{3}He decreases.

As T increases, importance of pp2 and pp3 relative to pp1 increases.

The proton-proton chain

pp1: $^{3}He + {}^{3}He \rightarrow {}^{4}He + {}^{1}H + {}^{1}H$ $\qquad r_{3,3} \sim \left({}^{3}He\right)^{2} \langle \sigma v \rangle_{3,3}$

pp2: $^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$ $\qquad r_{3,4} \sim {}^{3}He\,{}^{4}He \langle \sigma v \rangle_{3,4}$

pp2 will dominate when: $r_{3,4} > r_{3,3} \quad \rightarrow \quad \frac{{}^{4}He}{{}^{3}He} > \frac{\langle \sigma v \rangle_{3,3}}{\langle \sigma v \rangle_{3,4}} \gg 1$

pp2: $^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu_{e}$ $\qquad r_{7,-} \sim {}^{7}Be \times e^{-} \langle \sigma v \rangle_{7,-}$

pp3: $^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma$ $\qquad r_{7,+} \sim {}^{7}Be \times p^{+} \langle \sigma v \rangle_{7,+}$

pp3 will dominate when: $r_{7,+} > r_{7,-} \quad \rightarrow \quad \langle \sigma v \rangle_{7,+} > \langle \sigma v \rangle_{7,-}$

The proton-proton chain

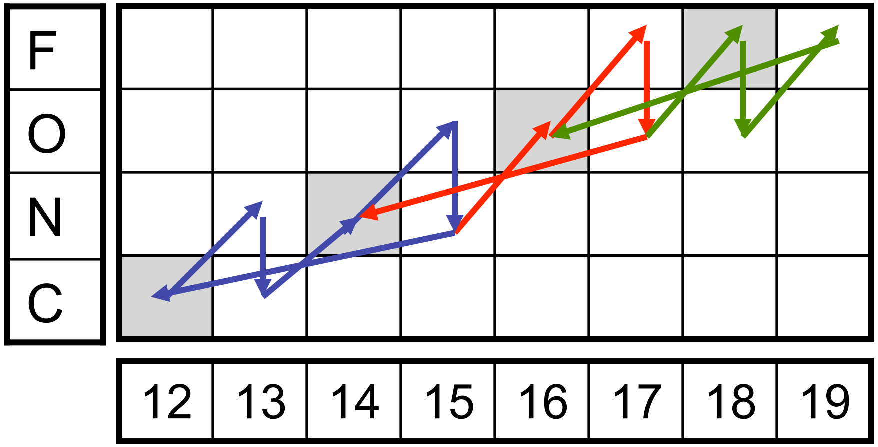
The energy generation rate is:

$$\varepsilon_{pp} = 2.4 \times 10^{4} \frac{\rho X^{2}}{T_{9}^{2/3}} \exp\left(-3.38/T_{9}^{1/3}\right) ergs^{-1} g^{-1}$$

The CNO cycle

CNO1

$$^{12}C + {}^{1}H \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^{+} + \nu_{e}$$
$$^{13}C + {}^{1}H \rightarrow {}^{14}N + \gamma$$
$$^{14}N + {}^{1}H \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_{e}$$
$$^{15}N + {}^{1}H \rightarrow {}^{12}C + {}^{4}He$$



CNO2

$$^{15}N + {}^{1}H \rightarrow {}^{16}O + \gamma$$
$$^{16}O + {}^{1}H \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^{+} + \nu_{e}$$
$$^{17}O + {}^{1}H \rightarrow {}^{14}N + {}^{4}He$$

CNO3

$$^{17}O + {}^{1}H \rightarrow {}^{18}F + \gamma$$
$$^{18}F \rightarrow {}^{18}O + e^{+} + \nu_{e}$$
$$^{18}O + {}^{1}H \rightarrow {}^{19}F + \gamma$$
$$^{19}F + {}^{1}H \rightarrow {}^{16}O + {}^{4}He$$

The CNO cycle

CNO1

$$^{12}C + {}^{1}H \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^{+} + \nu_{e}$$
$$^{13}C + {}^{1}H \rightarrow {}^{14}N + \gamma$$
$$^{14}N + {}^{1}H \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^{+} + \nu_{e}$$
$$^{15}N + {}^{1}H \rightarrow {}^{12}C + {}^{4}He$$

This is the slowest reaction and sets the overall rate.

After long enough time, the most abundant nucleus will be ^{14}N
Most CNO → ^{14}N

Timescale to reach equilibrium is long

CNO2

$$^{15}N + {}^{1}H \rightarrow {}^{16}O + \gamma$$
$$^{16}O + {}^{1}H \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^{+} + \nu_{e}$$
$$^{17}O + {}^{1}H \rightarrow {}^{14}N + {}^{4}He$$

CNO3

$$^{17}O + {}^{1}H \rightarrow {}^{18}F + \gamma$$
$$^{18}F \rightarrow {}^{18}O + e^{+} + \nu_{e}$$
$$^{18}O + {}^{1}H \rightarrow {}^{19}F + \gamma$$
$$^{19}F + {}^{1}H \rightarrow {}^{16}O + {}^{4}He$$

The CNO cycle

The approximate energy generation rate is:

$$\varepsilon_{CNO} \approx 4.4 \times 10^{25} \frac{\rho XZ}{T_9^{2/3}} \exp\left(-15.228/T_9^{1/3}\right) ergs^{-1}g^{-1}$$

The temperature sensitivity is:

$$\nu_{CNO} = \frac{50.8}{T_6^{1/3}} - \frac{2}{3}$$

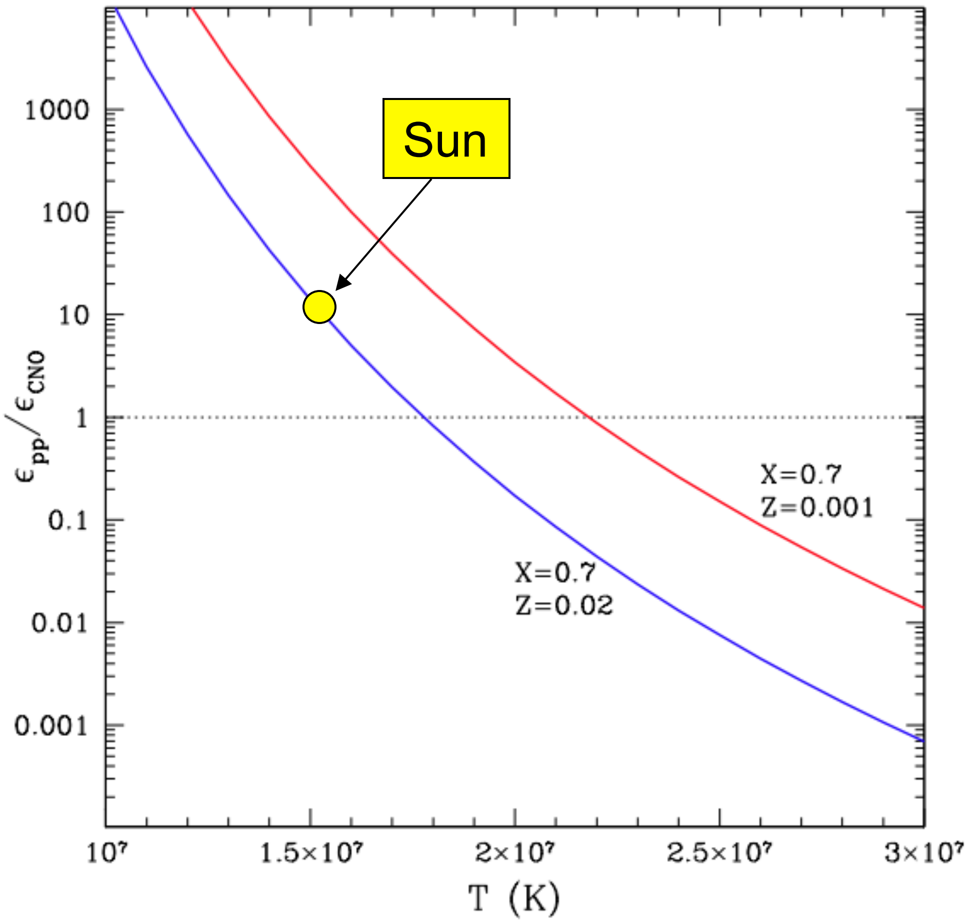
$$T_6 \approx 20 \rightarrow \nu_{CNO} \approx 18$$

pp chain vs. CNO cycle

CNO is favored over pp when:

- T is high
  Since heavier nuclei are involved

- Metallicity is high
  Since CNO nuclei are needed



He Burning (triple α)

$$^{4}He + ^{4}He \rightarrow ^{8}Be$$

This is the inverse of the final reaction in the pp3 chain.

The reaction is endothermic, absorbing 92 keV of energy. Requires about this much energy in Gamow peak

$$\frac{E_0}{kT} \approx 6.6 W^{1/3} T_7^{-1/3} \qquad W = Z_j^2 Z_k^2 \frac{A_j A_k}{A_j + A_k}$$

$$E_0 = 92 keV \rightarrow T = 1.15 \times 10^8 K$$

Lower limit for He burning

- When T>10^8K, ^{8}Be is created roughly as fast as it is destroyed (via inverse reaction)

We can find equilibrium abundance via $\langle \sigma v \rangle$, but we can also use a Saha equation for nuclei:

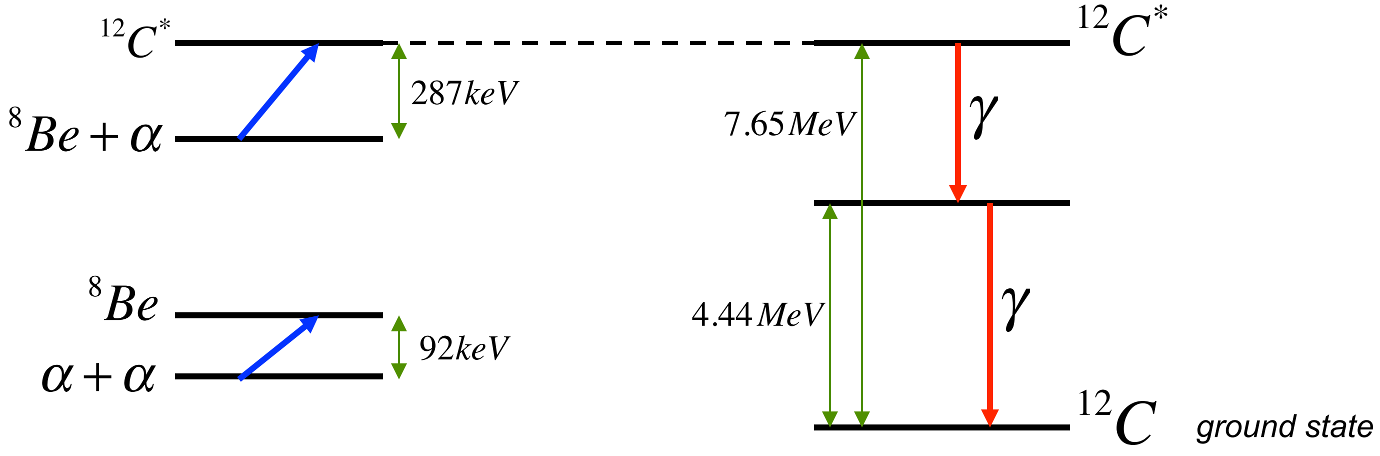
$$\frac{n_\alpha^2}{n_{^8Be}} = \left(\frac{\pi m_\alpha kT}{h^2} \right)^{3/2} e^{-Q/kT} \qquad Q = 92 keV \qquad T = 10^8 K \rightarrow \frac{n_{Be}}{n_\alpha} = 7 \times 10^{-9}$$

He Burning

$${}^{8}Be+{}^{4}He\rightarrow{}^{12}C+\gamma+\gamma$$

The reaction is exothermic and resonant, proceeding via an excited state ${}^{12}C^{*}$

Most ${}^{12}C^{*}$ decay straight back to ${}^{8}Be + {}^{4}He$, but some emit a photon to form a stable ${}^{12}C$



He Burning

The approximate energy generation rate is:

$$\varepsilon_{3\alpha} = 5\times10^{8}\frac{\rho^{2}Y^{3}}{T_9^{3}}\exp(-4.4/T_9)ergs^{-1}g^{-1}$$

The temperature sensitivity is:

$$\nu_{3\alpha} = \frac{4.4}{T_9} - 3$$

$$T_9 = 0.1 \rightarrow \nu_{3\alpha} \approx 40$$

He Burning

The effect of He burning on the structure of the star depends on the central conditions at the moment of ignition.

- Low mass stars ($M < 0.4M_{sun}$) develop a fully degenerate Helium core before the temperature rises to the He ignition threshold. The core cannot contract or heat up further → no He burning → He white dwarf

- High mass stars ($M > 1.5M_{sun}$) ignite He while the density is well below the degeneracy threshold (ρ<<10^6g cm^{-3}) → No large structural changes

- Intermediate mass stars ignite He under conditions of partial degeneracy. This allows a runaway reaction: He flash

He Flash

- Under degenerate conditions, pressure is a function of ρ only, not T.

- Upon He ignition, the energy release leads to a T rise. However, this does not lead to a P rise and hence to expansion and consequent reduction in ρ and T, as in a non-degenrate gas.

- Higher T increases the rate of nuclear reactions further → runaway

Non-degenerate core: He ignition → T ↑ → P ↑ → expand → T ↓ stable

Degenerate core: He ignition → T ↑ → ε ↑ → T ↑ unstable

- This continues for a few minutes until T is high enough in the core to lift degeneracy.
- During this time the core radiates with a luminosity similar to the whole Milky Way
- This energy is absorbed by the core and is not detected at the surface.
- The core then expands rapidly until a new stable He burning structure is attained.

Heavier Elements

- Once enough ^{12}C is formed by the triple-α reaction, further captures of α particles occur simultaneously with the Carbon forming reaction.

$$^{12}C+^{4}He\rightarrow^{16}O+\gamma$$

$$^{16}O+^{4}He\rightarrow^{20}Ne+\gamma$$

(Reactions beyond ^{20}Ne are rare.)

- He burning → C, O, Ne. If T>10^{9}K, further reactions can occur. Since binding energies of heavy nuclei are comparable, a wide range of reactions are possible.

$$^{12}C+^{12}C\rightarrow^{20}Ne+^{4}He$$

$$^{12}C+^{12}C\rightarrow^{23}Na+p$$

Most important C burning

$$^{16}O+^{16}O\rightarrow^{31}P+p$$

Most common O burning

$$^{16}O+^{16}O\rightarrow^{28}Si+^{4}He$$

$$^{16}O+^{16}O\rightarrow^{31}S+n$$

- Si burning starts at T~3x10^{9}K
- over long timescales → ^{56}Fe
- over short timescales → ^{56}Ni (*e.g., in SN*)