Summary of Results

1. Mass Conservation

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

eulerian

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

lagrangian

Summary of Results

2. Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

eulerian

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

lagrangian

Need equation of state: $P = f\left(\rho, T, X_i\right)$

Summary of Results

3. Energy Generation

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$$

eulerian

$$\frac{dL_m}{dm} = \varepsilon$$

lagrangian

Need nuclear physics: $\varepsilon = f\left(\rho, T, X_i\right)$

Summary of Results

4. Energy Flow: Radiation

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acr^2T^3}$$

eulerian

$$\frac{dT}{dm} = -\frac{3\kappa L_m}{64\pi^2 acr^4T^3}$$

lagrangian

Need opacity: $\kappa = f\left(\rho, T, X_i\right)$

Summary of Results

4. Energy Flow: Convection

$$\frac{dT}{dr} = -\left(1 - \frac{1}{\Gamma_2}\right)\frac{Gm\rho T}{r^2 P}$$

eulerian

$$\frac{dT}{dm} = -\left(1 - \frac{1}{\Gamma_2}\right)\frac{GmT}{4\pi r^4 P}$$

lagrangian

Need adiabatic exponent: $\Gamma_2 = f\left(\rho, T, X_i\right)$

Summary of Results

Equation of State (non-degenerate matter)

$$P = \frac{1}{3} a T^4 + \frac{N_A k}{\mu} \rho T$$

Mean molecular weight: $$\frac{1}{\mu} = \frac{1}{\mu_i} + \frac{1}{\mu_e} = \sum_i \frac{X_i}{A_i} + \sum_i \frac{Z_i X_i y_i}{A_i}$$

Summary of Results

Ionization fractions y_i

Saha equation :

$$\frac{n^+ n_e}{n^0} = \frac{2G^+}{G^0} \frac{\left(2\pi m_e kT\right)^{3/2}}{h^3} \exp\left(-\frac{\chi}{kT}\right)$$

e.g., Hydrogen only :

$$\frac{y^2}{1-y} = \frac{1}{N_A \rho} \frac{\left(2\pi m_e kT\right)^{3/2}}{h^3} \exp\left(-\frac{\chi_H}{kT}\right)$$

Summary of Results

Thermodynamics

Adiabatic index :

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad}$$

$$\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2} \qquad \beta = \frac{P_g}{P_{tot}}$$

(assuming neutral or fully ionized gas)

Summary of Results

Opacity: approximations

$$\frac{1}{\kappa} \approx \frac{1}{\kappa_{H^-}} + \frac{1}{\kappa_e + \kappa_{ff} + \kappa_{bf}}$$

$$\kappa_e = 0.2(1+X)\, cm^2 g^{-1}$$

(assuming fully ionized and no metals)

$$\kappa_{ff} \approx 4 \times 10^{22} (X+Y)(1+X) \rho T^{-3.5} cm^2 g^{-1}$$

$$\kappa_{bf} \approx 4 \times 10^{25} Z(1+X) \rho T^{-3.5} cm^2 g^{-1}$$

$$\kappa_{H^-} \approx 2.5 \times 10^{-31} (Z/0.02) \rho^{1/2} T^9 cm^2 g^{-1}$$

Summary of Results

Opacity: tables

Interpolate in density, temperature, and composition.

logR / logT	-6.0	-5.5
6.1	-0.429	-0.409
6.2	-0.429	-0.397

logT = 6.192

logR = -5.83

Summary of Results

Convection

Convection happens when: $$\left|\frac{dT}{dr}\right|_{rad} > \left|\frac{dT}{dr}\right|_{ad}$$

$$L_m > \frac{16\pi acG}{3\kappa_R}\left(1 - \frac{1}{\Gamma_2}\right)\frac{T^4 m}{P}$$

Summary of Results

Energy generation

$$\varepsilon = \varepsilon_{pp} + \varepsilon_{CNO} + \varepsilon_{3\alpha}$$

$$\varepsilon_{pp} = 2.4 \times 10^{4} \frac{\rho X^{2}}{T_9^{2/3}} \exp\left(-3.38/T_9^{1/3}\right) erg s^{-1} g^{-1}$$

$$\varepsilon_{CNO} \approx 4.4 \times 10^{25} \frac{\rho XZ}{T_9^{2/3}} \exp\left(-15.228/T_9^{1/3}\right) erg s^{-1} g^{-1}$$

$$\varepsilon_{3\alpha} = 5 \times 10^{8} \frac{\rho^{2} Y^{3}}{T_9^{3}} \exp(-4.4/T_9) erg s^{-1} g^{-1}$$

Stellar Models

Given M and X,Y,Z, solve structure equations

$$\frac{dr}{dm} \qquad \frac{dP}{dm} \qquad \frac{dT}{dm} \qquad \frac{dL_m}{dm}$$

to get r, P, T, L_m as a function of m

Need to know:

$$\rho(P,T,X,Y,Z) \qquad \varepsilon(\rho,T,X,Y,Z)$$

$$\kappa_R(\rho,T,X,Y,Z) \qquad \Gamma_2(\rho,T,X,Y,Z)$$

Stellar Models

CENTER

$$m = 0$$

$$r = 0 \quad ✓$$

$$P = P_c$$

$$T = T_c$$

$$L_m = 0 \quad ✓$$

SURFACE

$$m = M$$

$$r = R$$

$$P \approx 0 \quad ✓$$

$$T \approx 0 \quad ✓$$

$$L_m = L$$

Stellar Models

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dL_m}{dm} = \varepsilon$$

$$\frac{dT}{dm} = -\frac{3\kappa L_m}{64\pi^2 ac r^4 T^3}$$

$$\frac{dT}{dm} = -\left(1 - \frac{1}{\Gamma_2}\right)\frac{GmT}{4\pi r^4 P}$$

- Some of these equations are indeterminant at center.
- It is better to start the integration at a very small m>0.

Stellar Models

Central boundary conditions

At a very small m>0:

$$r \neq 0 \qquad m = \frac{4}{3}\pi r^3 \rho_c \rightarrow r = \left(\frac{3m}{4\pi\rho_c} \right)^{1/3}$$

$$L_m \neq 0 \qquad L_m = \varepsilon_c m$$

Stellar Models

Central boundary conditions

- We can get boundary conditions for P and T by expanding and demanding that their derivatives are zero at r=0

$$P(r) = P(0) + P'(0)(r-0) + \frac{1}{2}P''(0)(r-0)^2$$

0

$$= P_c + P''(0)\frac{r^2}{2}$$

$$P(r) = P_c - \frac{2}{3}\pi G \rho_c^2 r^2$$

$$P'(r) = -\frac{Gm\rho}{r^2}$$

$$= -\frac{G\rho}{r^2}\left(\frac{4}{3}\pi r^3 \rho_c\right) = -\frac{4}{3}\pi G \rho_c^2 r$$

Stellar Models

Central boundary conditions

- Similarly,

$$T(r) = T(0) + T'(0)(r-0) + \frac{1}{2}T''(0)(r-0)^2$$

0

$$= T_c + T''(0)\frac{r^2}{2}$$

$$T'(r) = -\frac{3\kappa_R \rho L_r}{16\pi a c r^2 T^3}$$

$$T'(r) = \left(1 - \frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$$

Radiative

$$T(r) = T_c - \frac{\kappa_c \rho_c^2 \varepsilon_c}{8acT_c^3} r^2$$

Convective

$$T(r) = T_c - \left(1 - \frac{1}{\Gamma_{2,c}}\right)\frac{2\pi G \rho_c^2 T_c}{3P_c} r^2$$

Stellar Models

Surface boundary conditions

- We can do better than T=P=0 at surface.

guess R and L:

$$L = 4\pi R^2 \sigma T_s^4 \rightarrow \qquad T_s = \left(\frac{L}{4\pi R^2 \sigma} \right)^{1/4}$$

Guess ρ very small, e.g.,. 10-5 g cm-3:

$$P_s = \frac{1}{3} a T_s^4 + \frac{N_A k}{\mu} \rho_s T_s$$

Stellar Models

Steps in constructing a stellar model

1. Compute $\rho, \kappa, \varepsilon, \Gamma_2$ as a function of P, T, X, Y, Z

DENSITY

- Beware of P dropping below $\frac{1}{3}aT^4$

OPACITY

- Use approximations for crude results
- Interpolate using tables (use approximations outside table bounds)

ENERGY GENERATION

- Use formulas for pp, CNO, and 3 α

THERMODYNAMICS

- $\Gamma_2 =$ 5/3 for ideal gas, 4/3 for radiation pressure
- Γ_2 departs from these values when
  - Mixture of ideal gas and radiation
  - Ionization zones

Stellar Models

Steps in constructing a stellar model

2. Use four structure equations to compute $r,\ P,\ T,\ L_m$ vs. m given starting values for these (boundary conditions).

- Given values of $m,\ r,\ P,\ T,\ L_m,\ \rho,\ \kappa,\ \varepsilon,\ \Gamma_2$ at shell i compute 4 derivatives: $\frac{dr}{dm}, \frac{dP}{dm}, \frac{dL_m}{dm}, \frac{dT}{dm}$

- Use these derivatives to compute $r,\ P,\ T,\ L_m$ at shell $i+1$

  e.g.,

$$r[i+1] = r[i] + \left(\frac{dr}{dm}\right)[i] \times dm$$

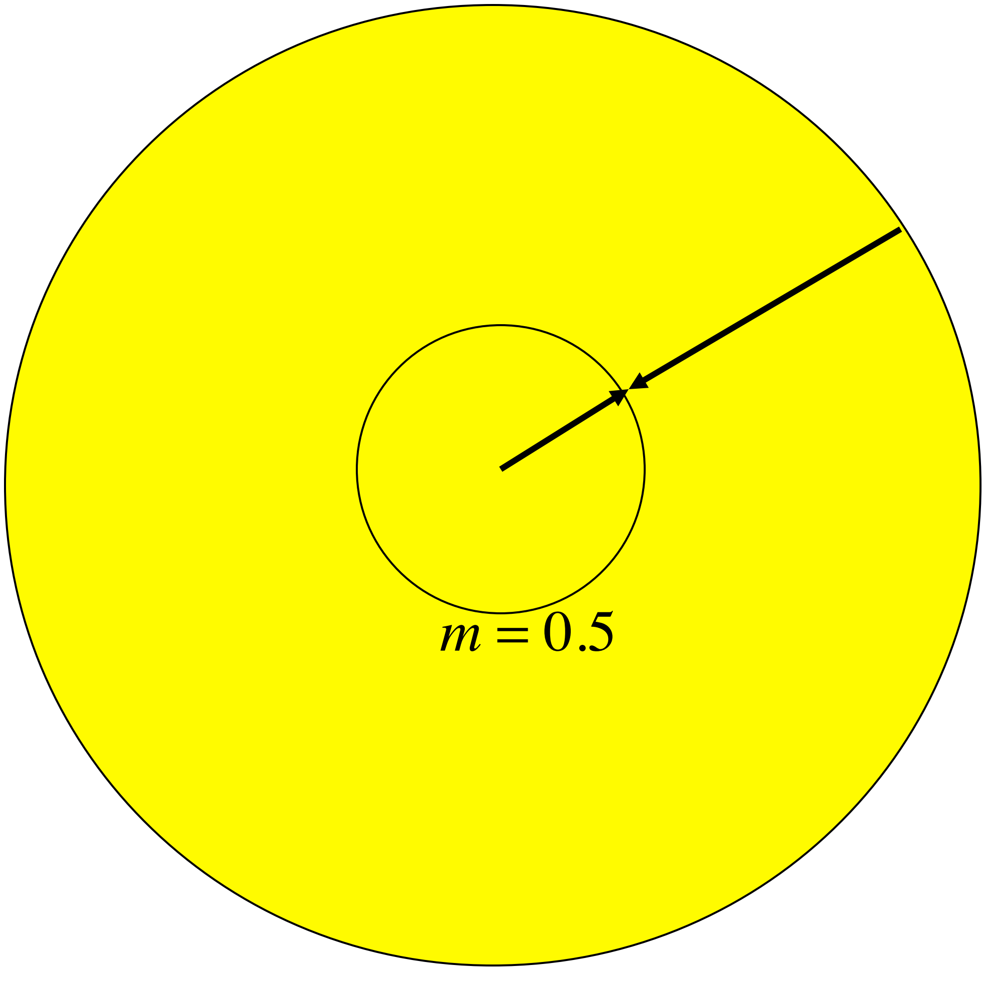
- Shell size $dm < 10^{-4}\,M$ near center and surface.

Stellar Models

Steps in constructing a stellar model

3. Deal with lack of complete boundary conditions at center or surface.
   - At center, have r, L_m, but not P, T
   - At surface, have P, T, but not r, L_m
   - General approach: guess values for missing conditions at one end, run model, and compare boundary conditions at other end.
     PROBLEM: small changes in conditions at center can cause large differences at surface → difficult to reach convergence.
     SOLUTION: Shoot from both center and surface and meet halfway through star.

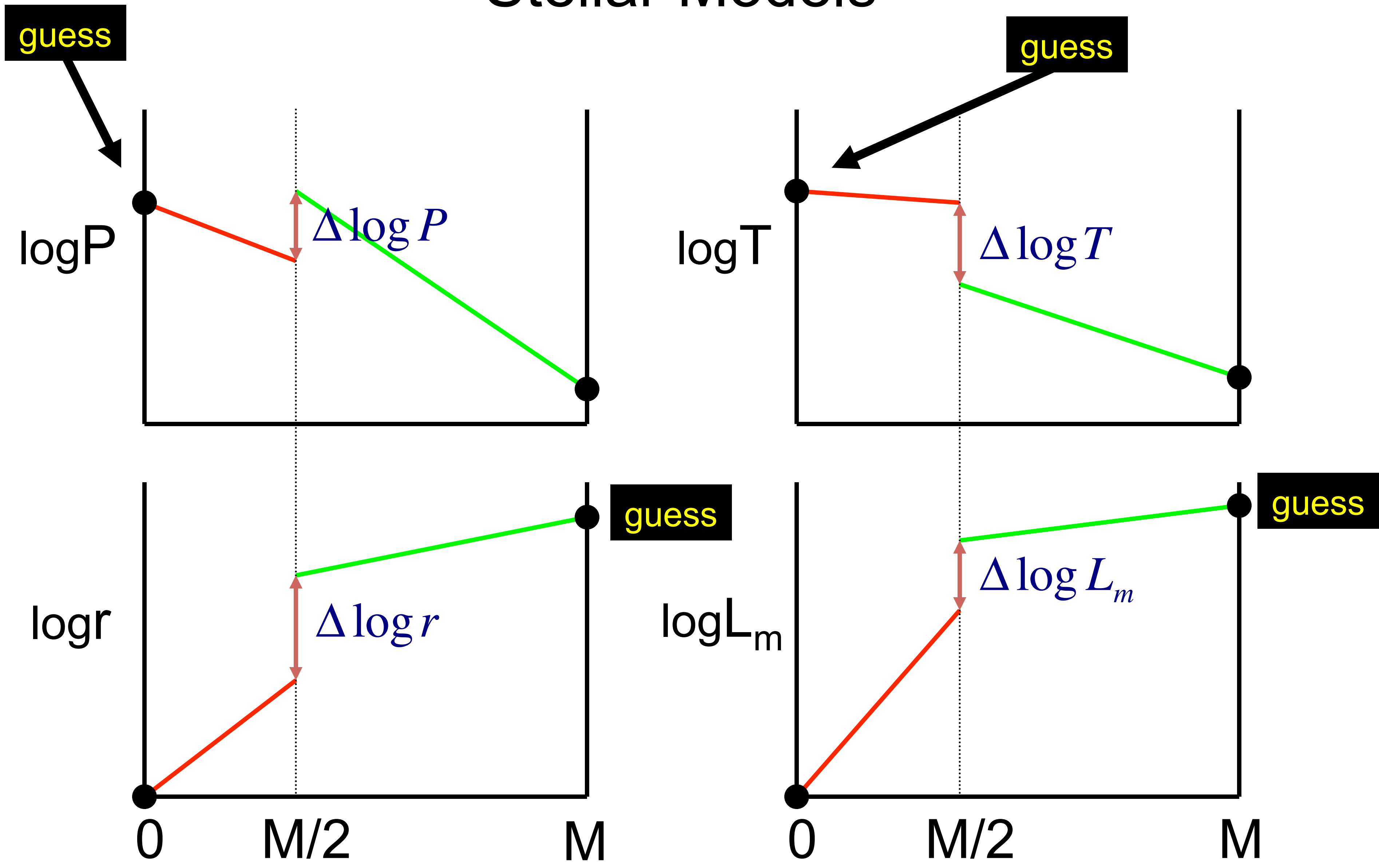
Stellar Models



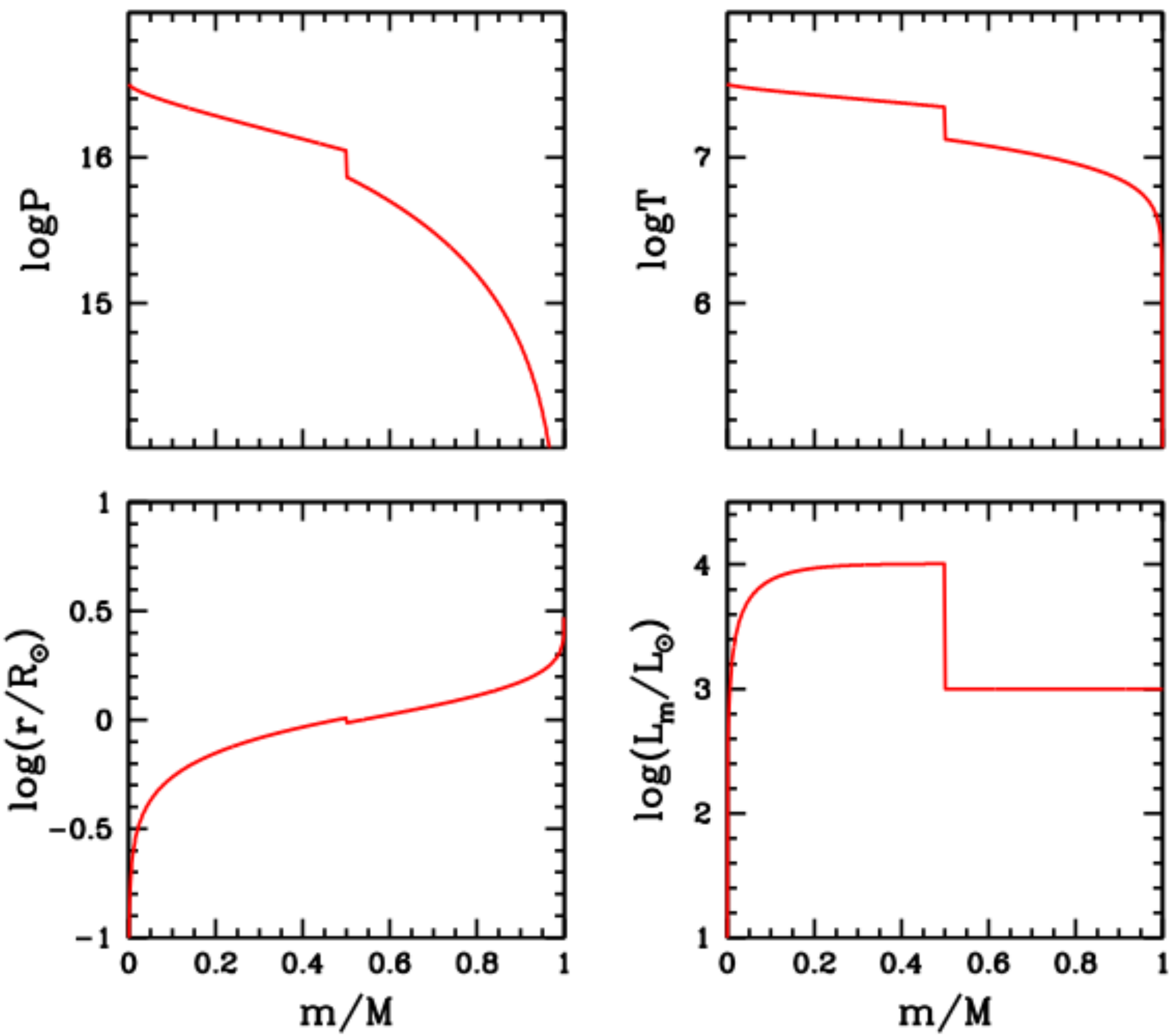
Stellar Models

- Guess values for T_c and P_c and integrate outwards from $m = 0$ to $m = M/2$
- Guess values for R and L and integrate inwards from $m = M$ to $m = M/2$
- Compute discrepancies $\Delta r,\ \Delta P,\ \Delta T,\ \Delta L_m$ at $m = M/2$
- Work in log space: $\log P,\ \log T,\ \log r,\ \log L_m$

Stellar Models



Stellar Models



Stellar Models

Boundary Conditions

$\log P_c$

$\log R$

$\log T_c$

$\log L$

$\Delta \log P$

$\Delta \log T$

$\Delta \log r$

$\Delta \log L_m$

Stellar Models

- Repeat using new trial values:

$$\left(\log T_c - d\log T_c\right) \text{ and } \left(\log T_c + d\log T_c\right) \text{ with } \log P_c,\ \log R,\ \log L$$

$$\left(\log P_c - d\log P_c\right) \text{ and } \left(\log P_c + d\log P_c\right) \text{ with } \log T_c,\ \log R,\ \log L$$

$$\left(\log R - d\log R\right) \text{ and } \left(\log R + d\log R\right) \text{ with } \log P_c,\ \log T_c,\ \log L$$

$$\left(\log L - d\log L\right) \text{ and } \left(\log L + d\log L\right) \text{ with } \log P_c,\ \log T_c,\ \log R$$

- Compute new discrepancies in each case.

$$\Delta \log r,\ \Delta \log P,\ \Delta \log T,\ \Delta \log L_m$$

- Get 16 derivatives. e.g.,

$$\frac{\partial(\Delta \log r)}{\partial \log P_c} = \frac{(\Delta \log r)_{\log P_c + d\log P_c} - (\Delta \log r)_{\log P - d\log P_c}}{2d\log P_c}$$

Stellar Models

- Use these derivatives to calculate improved boundary conditions.

$$\log P_c' = \log P_c + \delta \log P_c$$
$$\log T_c' = \log T_c + \delta \log T_c$$
$$\log R' = \log R + \delta \log R$$
$$\log L' = \log L + \delta \log L$$

- General idea:

$$-\Delta \log r = \delta \log P_c \frac{\partial(\Delta \log r)}{\partial \log P_c} \rightarrow \delta \log P_c = -\Delta \log r \left[\frac{\partial(\Delta \log r)}{\partial \log P_c} \right]^{-1}$$

- More complicated with 4 variables!

Stellar Models

$$
\begin{pmatrix}
\frac{\partial(\Delta \log P)}{\partial \log P_c} & \frac{\partial(\Delta \log P)}{\partial \log T_c} & \frac{\partial(\Delta \log P)}{\partial \log R} & \frac{\partial(\Delta \log P)}{\partial \log L} \\
\frac{\partial(\Delta \log T)}{\partial \log P_c} & \frac{\partial(\Delta \log T)}{\partial \log T_c} & \frac{\partial(\Delta \log T)}{\partial \log R} & \frac{\partial(\Delta \log T)}{\partial \log L} \\
\frac{\partial(\Delta \log r)}{\partial \log P_c} & \frac{\partial(\Delta \log r)}{\partial \log T_c} & \frac{\partial(\Delta \log r)}{\partial \log R} & \frac{\partial(\Delta \log r)}{\partial \log L} \\
\frac{\partial(\Delta \log L_m)}{\partial \log P_c} & \frac{\partial(\Delta \log L_m)}{\partial \log T_c} & \frac{\partial(\Delta \log L_m)}{\partial \log R} & \frac{\partial(\Delta \log L_m)}{\partial \log L}
\end{pmatrix}
\times
\begin{pmatrix}
\delta \log P_c \\
\delta \log T_c \\
\delta \log R \\
\delta \log L
\end{pmatrix}
=
\begin{pmatrix}
-\Delta \log P \\
-\Delta \log T \\
-\Delta \log r \\
-\Delta \log L_m
\end{pmatrix}
$$

- Invert matrix to solve for $\delta \log P_c,\ \delta \log T_c,\ \delta \log R,\ \delta \log L$
  (actually take smaller steps ~0.1x)

- Iterate until convergence is reached: discrepancies vanish (i.e., drop below a threshold value)