Global Energetics

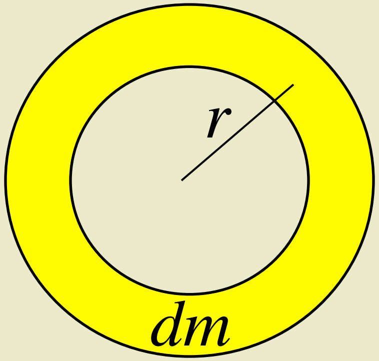
Total stellar energy

$$W = \text{gravitational potential energy } \Omega$$
$$+ \text{ kinetic energy due to bulk motions (turbulence, pulsation)}$$
$$+ \text{ internal energy from microscopic processes } U$$

Gravitational Potential Energy

<u>Gravitational potential energy</u>

Ω : energy required to assemble the star by collecting material from the outside universe.
= negative of energy required to disperse the star.



$$d\Omega = -\int_r^\infty \frac{Gmdm}{r'^2}dr' = -\frac{Gm}{r}dm$$

$$\Omega = -\int_0^M \frac{Gm}{r}dm$$

which depends on the density profile $\rho(r)$

Gravitational Potential Energy

e.g., for $\rho(r) = \rho_0$

$$\left.\begin{aligned} m &= \rho_0 \frac{4}{3}\pi r^3 \\ M &= \rho_0 \frac{4}{3}\pi R^3 \end{aligned}\right\} \rightarrow \quad \frac{m}{M} = \left(\frac{r}{R}\right)^3 \rightarrow \quad r = R\left(\frac{m}{M}\right)^{\frac{1}{3}}$$

$$\Omega = -\int_0^M \frac{Gm}{r}\,dm = -\frac{GM^{1/3}}{R}\int_0^M m^{2/3}\,dm \qquad = -\frac{3}{5}\frac{GM^2}{R}$$

In general, $\Omega = -q\dfrac{GM^2}{R}$ and $q > \dfrac{3}{5}$ since $\rho \downarrow$ with r

$$\rho \sim r^{-1} \rightarrow \quad q = \frac{2}{3} \qquad\qquad \rho \sim r^{-2} \rightarrow \quad q = 1$$

Internal and Kinetic Energy

Internal energy

U : Kinetic energy due to motions of particles (thermal, not bulk)
+ energy in atomic and molecular bonds.

$$U = -\int_M E_m \, dm$$ Where E_m : local specific internal energy (erg/g)

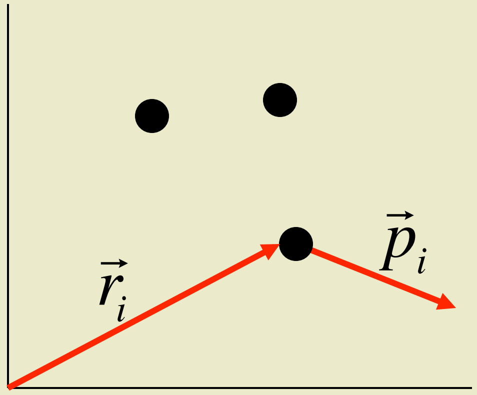
Kinetic energy

K : Total kinetic energy (thermal + bulk)

The Virial Theorem

Consider all particles in a star at positions $\vec{r}_i$ with momenta $\vec{p}_i$

$$Q = \sum_i \vec{p}_i \cdot \vec{r}_i$$



evaluate $\dfrac{dQ}{dt}$

what are the units of $\dfrac{dQ}{dt}$?

(energy units: $\dfrac{m \cdot v \cdot r}{t} = m \cdot v^2$)

The Virial Theorem

$$Q = \sum_i \vec{p}_i \cdot \vec{r}_i$$

$$\frac{dQ}{dt} = \frac{d}{dt}\sum_i m \cdot \dot{\vec{r}}_i \cdot \vec{r}_i$$

$$= \frac{d}{dt}\sum_i \frac{d}{dt}\left(\frac{1}{2} m \cdot \vec{r}_i^{\,2}\right)$$

$$= \frac{1}{2}\sum_i \frac{d^2}{dt^2}\left(m \cdot \vec{r}_i^{\,2}\right)$$

$$\sum_i m \cdot \vec{r}_i^{\,2}$$

is just the total moment of inertia of the system.

$$\frac{dQ}{dt} = \frac{1}{2}\frac{d^2 I}{dt^2}$$

The Virial Theorem

$$Q = \sum_i \vec{p}_i \cdot \vec{r}_i$$

$$\frac{dQ}{dt} = \underbrace{\sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i}_{\text{A}} + \underbrace{\sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}}_{\text{B}}$$

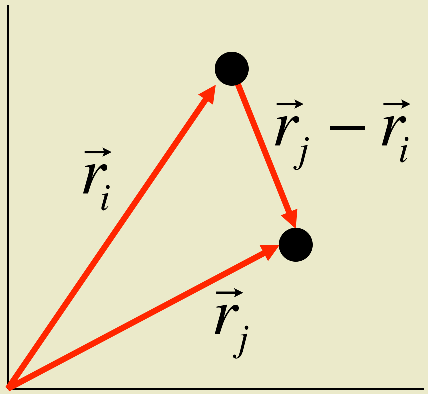
B: $\sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt} = \sum_i m \cdot \vec{v}_i \cdot \vec{v}_i$

$= 2\sum_i \frac{1}{2} m \cdot \vec{v}_i^{\,2}$ $= 2K$ (2 x total kinetic energy)

The Virial Theorem

A: $$\sum_i \frac{d\vec{p}_i}{dt}\cdot\vec{r}_i = \sum_i \frac{d}{dt}\left(m\cdot\dot{\vec{r}}_i\right)\cdot\vec{r} = \sum_i m\cdot\ddot{\vec{r}}_i\cdot\vec{r}_i = \sum_i \vec{F}_i\cdot\vec{r}_i$$

$\vec{F}_i$ is the sum of all gravitational forces on particle *i* $= \sum_{j\neq i}\vec{F}_{ij}$



$$\vec{F}_{ij} = G\frac{m_i m_j}{r_{ij}^2}\frac{\left(\vec{r}_j-\vec{r}_i\right)}{r_{ij}} = -G\frac{m_i m_j}{r_{ij}^3}\left(\vec{r}_i-\vec{r}_j\right)$$

$$\sum_i \vec{F}_i\cdot\vec{r}_i = \sum_i\sum_{j\neq i}\vec{F}_{ij}\cdot\vec{r}_i = \cdots+\vec{F}_{ij}\cdot\vec{r}_i+\vec{F}_{ji}\cdot\vec{r}_j+\cdots$$

$$\left(\vec{F}_{ij}=-\vec{F}_{ji}\right) \qquad = \cdots+\vec{F}_{ij}\cdot\left(\vec{r}_i-\vec{r}_j\right)+\cdots = \sum_i\sum_{j>i}\vec{F}_{ij}\cdot\left(\vec{r}_i-\vec{r}_j\right)$$

$$= \sum_i\sum_{j>i}-G\frac{m_i m_j}{r_{ij}^3}\cdot\left(\vec{r}_i-\vec{r}_j\right)^2 = \sum_i\sum_{j>i}-G\frac{m_i m_j}{r_{ij}} = \Omega$$

gravitational potential energy

The Virial Theorem

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega$$

In a globally static system: $\frac{d^2 I}{dt^2} = 0$ → $2K = -\Omega$

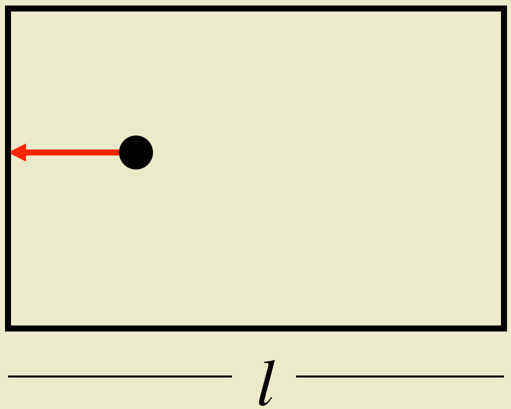
For a collisionless system: $W = K + \Omega$ → $W = \frac{1}{2}\Omega = -K$

bound!

What about a collisional system? How does the $2K$ term relate to the internal energy of a gas?

The Virial Theorem

<u>Kinetic theory</u>: pressure in a gas is caused by the sum of collisions of particles and momentum transfer



Particle *i* colliding with wall in x-direction

$$\Delta p_i = 2p_{x,i} = 2m_i v_{x,i}$$

Time between collisions of same particle with wall

$$\Delta t = \frac{2l}{v_{x,i}}$$

Force due to particle i

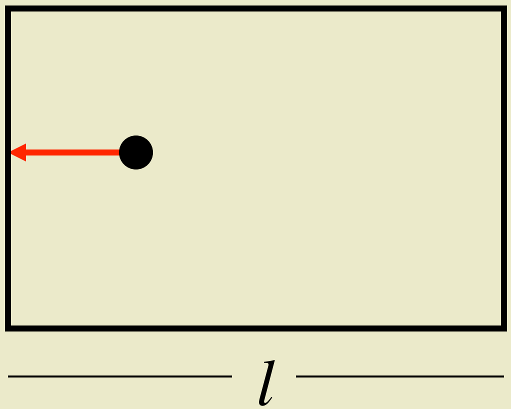
$$F_i = \frac{\Delta p_i}{\Delta t} = \frac{m_i v_{x,i}^2}{l}$$

Total force on wall

$$F = \sum_i \frac{m_i v_{x,i}^2}{l} = \frac{1}{l}\sum_i m_i v_{x,i}^2$$

The Virial Theorem

<u>Kinetic theory</u>: pressure in a gas is caused by the sum of collisions of particles and momentum transfer



Total force on all 6 walls

$$F = \frac{2}{l}\sum_i m_i\left(v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2\right) = \frac{2}{l}\sum_i m_i v_i^2$$

Total force on 1 wall

$$F = \frac{1}{3l}\sum_i m_i v_i^2 \quad = \frac{2}{3l}K$$

Pressure

$$P = \frac{F}{A} = \frac{2K}{3l \cdot A} = \frac{2K}{3V} \longrightarrow \boxed{2K = 3P \cdot V}$$

The Virial Theorem

In reality, pressure is changing over volume
(can think of our box as a volume element *dV*)

$$2K = 3\int_V P dV$$

In a star's spherical shells

$$dm = \rho dV$$

$$2K = \int_M \frac{3P}{\rho} dm$$

Depends on the equation of state

For a relation btw *P* and E_m of $P = (\gamma - 1)\rho E_m$ (γ-law equation of state)

$$2K = \int_M 3(\gamma - 1) E_m \, dm = 3(\gamma - 1)\int_M E_m \, dm = 3(\gamma - 1)U$$

K is only equal to U if $\gamma = \frac{5}{3}$ ⟶ i.e., for an ideal monoatomic gas

The Virial Theorem

Virial theorem: $3(\gamma - 1)U + \Omega = 0$

Total energy

$$W = \Omega + U$$
$$= \Omega - \frac{\Omega}{3(\gamma - 1)}$$
$$= \Omega\left(1 - \frac{1}{3(\gamma - 1)}\right)$$
$$= \Omega \frac{3\gamma - 4}{3(\gamma - 1)}$$

Total energy must be < 0 for star to be stable. Otherwise, it has enough energy to disperse itself.

$$\rightarrow \gamma > \frac{4}{3}$$

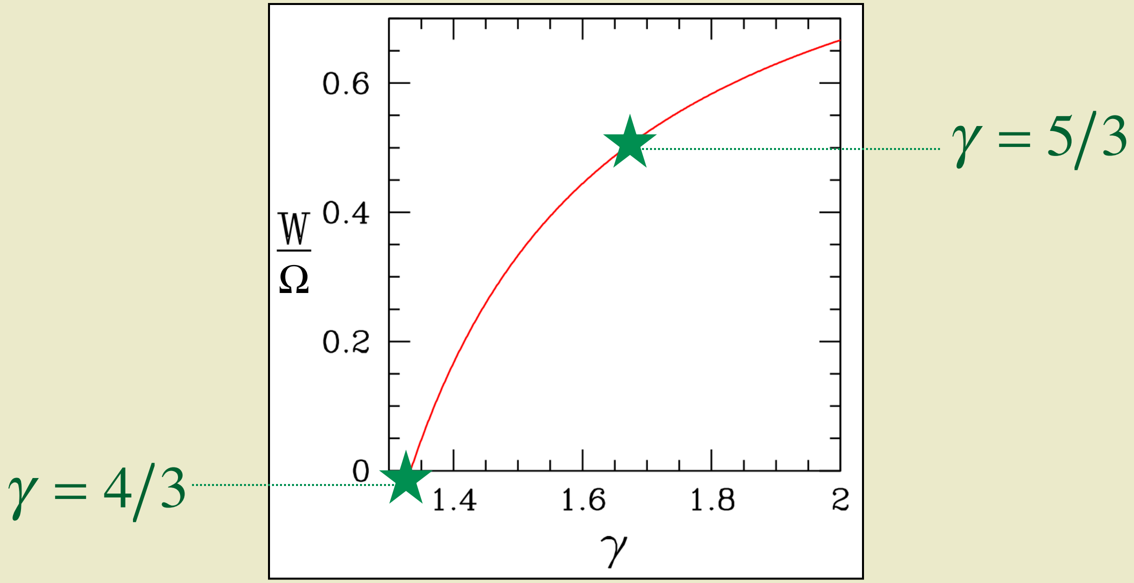
- Ideal gas $\left(\gamma = \frac{5}{3}\right)$: safely bound
- Radiation $\left(\gamma = \frac{4}{3}\right)$: unstable

As radiation pressure becomes more important, star becomes less bound.

The Virial Theorem

Virial theorem: $3(\gamma - 1)U + \Omega = 0$

$$W = \Omega \frac{3\gamma - 4}{3(\gamma - 1)}$$



As a star radiates, it loses energy (*without other energy sources*).

$$\dot{W} < 0 \rightarrow \begin{cases} \dot{\Omega} < 0 \\ \dot{U} > 0 \end{cases} \longrightarrow$$ star contracts and heats up

The Virial Theorem: Applications

Internal temperature

$$3(\gamma - 1)U + \Omega = 0$$

- For an ideal monoatomic gas: $\gamma = \frac{5}{3} \;\rightarrow\; U = -\frac{\Omega}{2}$

- Also, the energy density is: $E = \frac{3}{2}nkT \;\rightarrow\; U = \frac{3}{2}nkTV$

$$\left.\begin{array}{lll} n & : & \text{number density of particles} \\ \mu & : & \text{mean molecular weight} \\ N_A & : & \text{Avogadro's number} \end{array}\right\} n = \frac{\rho N_A}{\mu}$$

$$U = \frac{3}{2}\frac{\rho N_A}{\mu}kTV \;=\; \frac{3}{2}\frac{N_A}{\mu}kMT \;\rightarrow\; T = \frac{2}{3}\frac{\mu}{N_A k}\frac{U}{M}$$

- Virial theorem: $U = -\frac{\Omega}{2} \;=\; \frac{q}{2}\frac{GM^2}{R}$

The Virial Theorem: Applications

Internal temperature

$$T = \frac{q}{3}\frac{\mu G}{N_A k}\frac{M}{R} \longrightarrow T = 5\times10^6 K\left(\frac{q}{3/5}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}$$

(actual central temperature of sun is 15 million K)

- Internal T >> surface T ➔ strong T gradient
- Most of the sun is highly ionized
- T is sufficiently high for nuclear fusion (~1 million K)
- Fusion happens due to gravitational energy

Timescales

Dynamical timescale

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega$$

- Virial theorem: $\dfrac{d^2 I}{dt^2} \approx \dfrac{GM^2}{R}$

$$\left.\begin{array}{l} \dfrac{I}{t_{\rm dyn}^2} \approx \dfrac{GM^2}{R} \quad \rightarrow \quad t_{\rm dyn} \approx \left(\dfrac{R \cdot I}{GM^2}\right)^{1/2} \\ I \approx MR^2 \end{array}\right\} \rightarrow t_{\rm dyn} \approx \left(\frac{R^3}{GM}\right)^{1/2} \approx (G\rho)^{-1/2}$$

- Escape velocity: $v_{\rm esc}^2 \sim \dfrac{GM}{R}$
- Timescale for collapse: $t_{\rm dyn} \sim \dfrac{R}{v_{\rm esc}} \sim \dfrac{R}{\left(\dfrac{GM}{R}\right)^{1/2}}$

$$t_{\rm dyn} \sim (G\rho)^{-1/2}$$

For sun, $t_{\rm dyn}$~ 1 hour

Timescales

Kelvin-Helmholtz timescale

Timescale for star to radiate away its thermal/gravitational energy.

- Star contracts slowly maintaining hydrostatic equilibrium

$$\Delta R < 0 \quad \Downarrow \quad \Delta\Omega < 0$$

$$\left.\begin{aligned} \Omega &= -q\frac{GM^2}{R} \\ \frac{d\Omega}{dR} &= q\frac{GM^2}{R^2} \end{aligned}\right\} \rightarrow \Delta\Omega = +q\frac{GM^2}{R^2}\Delta R$$

- Virial theorem: $\gamma = \frac{5}{3} \rightarrow W = \frac{\Omega}{2} \rightarrow \Delta W < 0$

- Energy is lost from the system: radiation

$$\Delta W = \frac{\Delta\Omega}{2}$$

half goes to increasing *U* and half goes to luminosity

Timescales

Kelvin-Helmholtz timescale

Timescale for star to radiate away its thermal/gravitational energy.

- Suppose contraction is solely responsible for maintaining luminosity

$$L = -\frac{dW}{dt} = -\frac{d}{dt}\left(\frac{\Omega}{2}\right) = -\frac{1}{2}\frac{d\Omega}{dR}\frac{dR}{dt} = -\frac{q}{2}\frac{GM^2}{R^2}\frac{dR}{dt}$$

$$\frac{dR}{dt} \approx -\frac{R}{t_{\mathrm{KH}}}$$

$$t_{\mathrm{KH}} = \frac{q}{2}\frac{GM^2}{LR}$$

$$t_{\mathrm{KH}} = 2\times 10^7\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^2\left(\frac{L}{L_\odot}\right)^{-1}\left(\frac{R}{R_\odot}\right)^{-1}$$

- Too short a timescale to have happened.

Timescales

Nuclear timescale

Timescale for star to burn its H fuel and leave the Main Sequence.

$$t_{\rm nuc} = \frac{\text{total available energy}}{\text{luminosity}}$$

$$t_{\rm nuc} = \frac{\varepsilon f_{\rm core} M c^2}{L}$$

ε = efficiency of nuclear burning ($4\,{}^1H \rightarrow {}^4He$) ~ 0.007

$f_{\rm core}$ = mass fraction of core (star leaves main sequence after core burns) ~ 0.1

For sun, $t_{\rm nuc}$ ~ 10 billion years

For stars of solar mass or greater: $$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5}$$

$$t_{\rm nuc} \approx 10^{10}\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^{-2.5}$$

Timescales

$$t_{\mathrm{dyn}} \sim 1 \text{ hour}$$

$$t_{\mathrm{KH}} \sim 20 \text{ million years}$$

$$t_{\mathrm{nuc}} \sim 10 \text{ billion years}$$

$$t_{\mathrm{dyn}} \ll t_{\mathrm{KH}} \ll t_{\mathrm{nuc}}$$

It is valid to assume that stars on the main sequence are in dynamical and thermal equilibrium.