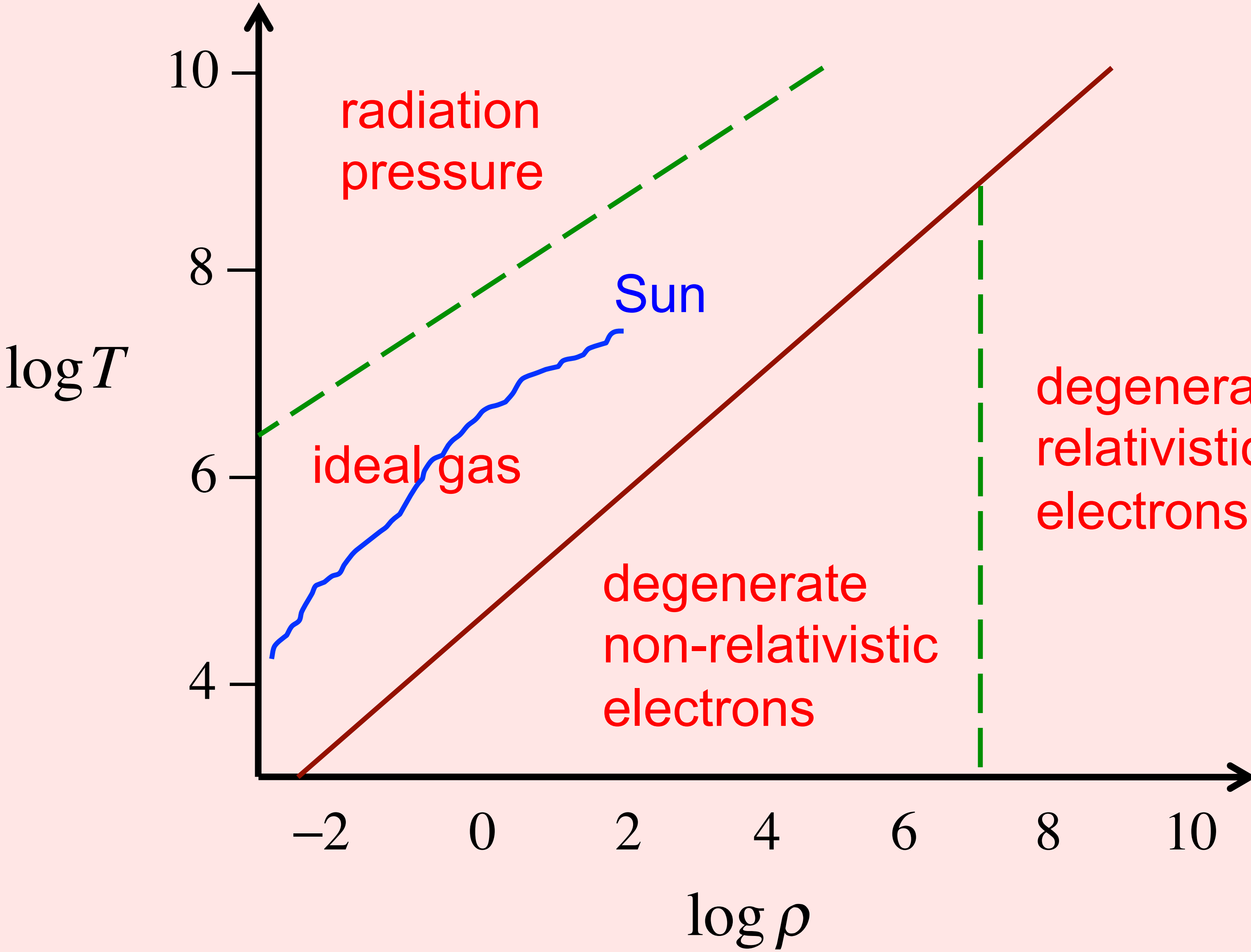
Equation of State

$$P = f\left(\rho, T, X_i\right)$$

To derive the E.O.S., we need to consider:

- Quantum statistics (bosons vs. fermions)
- Non-relativistic vs. relativistic particles
- Degenerate vs. non-degenerate matter

Equation of State



Equation of State

Radiation pressure and degenerate relativistic electrons

$$\gamma = \frac{4}{3} \quad \text{(star is not bound)}$$

This sets limits on the masses of stars

For most normal stars, ideal gas + radiation pressure are the important contributions

$$P = P_{\text{gas}} + P_{\text{rad}} \qquad = nkT + \frac{1}{3}aT^4$$

n is the number density of all atoms and free electrons

Mean Molecular Weight

To calculate the number density *n*, we need to know:

- constituents of plasma
- state of ionization

Available particles: ions + free electrons

$$n = n_I + n_e = \sum_i \left(n_{I,i} + n_{e,i} \right)$$

Each species *i* :

A_i : nuclear mass number

Z_i : nuclear charge

X_i : fraction by mass

e.g., $A_{^1H} = 1,\ A_{^4He} = 4,\ A_{^{12}C} = 12$

usually $Z_i = \dfrac{A_i}{2}$ for metals

Mean Molecular Weight

<u>1 mole</u> = amount that has as many atoms of substance as there are atoms of ^{12}C in $12g$ of ^{12}C.

Avogadro's number: $N_A = 6 \times 10^{23}$

$$\text{e.g.,}\quad \left.\begin{array}{llll} \text{1 mole of} & ^{12}C & \text{weighs} & 12g \\ \text{1 mole of} & ^{1}H & \text{weighs} & 1g \\ \text{1 mole of} & ^{4}He & \text{weighs} & 4g \end{array}\right\} \approx A_i$$

$$\frac{A_i}{N_A} = \frac{\text{mass of mole}}{\text{\# of atoms per mole}} = \text{mass per atom} \quad \text{(in grams)}$$

Mean Molecular Weight

Ions

$$n_{I,i} = \frac{(\text{mass/unit volume})\ \text{of}\ i}{(\text{mass of 1 ion})\ \text{of}\ i} = \frac{\rho X_i}{A_i / N_A} = \frac{\rho N_A X_i}{A_i}$$

$$n_I = \sum_i n_{I,i} = \rho N_A \sum_i \frac{X_i}{A_i}$$

$$\frac{\mu_I}{N_A} = \frac{\text{mass of mole for a mixture of ions}}{\#\ \text{of atoms per mole}} = \text{mean mass per atom}$$

$$\boxed{n_I = \frac{\rho N_A}{\mu_I}} \qquad \boxed{\mu_I^{-1} = \sum_i \frac{X_i}{A_i}}$$

Mean Molecular Weight

electrons

also need to know ionization fraction y_i

$$n_{e,i} = n_{I,i} \cdot Z_i \cdot y_i \quad = \rho N_A \frac{X_i}{A_i} Z_i y_i$$

$$y_i = \begin{cases} 0: & \text{completely neutral} \\ 1: & \text{completely ionized} \end{cases}$$

$$n_e = \sum_i n_{e,i} = \rho N_A \sum_i \frac{X_i Z_i y_i}{A_i}$$

$$\boxed{n_e = \frac{\rho N_A}{\mu_e}}$$

$$\boxed{\mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}}$$

Mixture of ions + electrons

μ is the mean weight of particles in units of ^{1}H

$$\boxed{n = \frac{\rho N_A}{\mu}}$$

$$\boxed{\mu^{-1} = \mu_I^{-1} + \mu_e^{-1}}$$

Mean Molecular Weight

$$\left({}^{1}H,\, {}^{4}He,\, \text{metals}\right) \equiv \left(X, Y, Z\right)$$

Composition of Sun

$$
\begin{aligned}
{}^{1}H &\sim 0.71 \\
{}^{4}He &\sim 0.27 \\
{}^{16}O &\sim 0.01 \\
{}^{12}C &\sim 0.004 \\
{}^{56}Fe &\sim 0.0015 \\
{}^{28}Si &\sim 0.001 \\
{}^{14}N &\sim 0.001 \\
{}^{24}Mg &\sim 0.0008 \\
{}^{20}Ne &\sim 0.0006 \\
{}^{32}S &\sim 0.0004
\end{aligned}
$$

These were fused inside stars

In the core

$$
\begin{aligned}
{}^{1}H &\sim 0.34 \\
{}^{4}He &\sim 0.64
\end{aligned}
$$

Mean Molecular Weight

$$\mu_I^{-1} = \sum_i \frac{X_i}{A_i}$$

$$\mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}$$

$$\mu^{-1} = \mu_I^{-1} + \mu_e^{-1}$$

- Fully ionized gas with zero metals (Z=0, y_i=1)

$$\mu_I^{-1} = \frac{X}{A_H} + \frac{Y}{A_{He}} = \frac{X}{1} + \frac{Y}{4} = X + \frac{1-X}{4} \boxed{= \frac{1+3X}{4}}$$

$$\mu_e^{-1} = \frac{X \cdot Z_H \cdot y_H}{A_H} + \frac{Y \cdot Z_{He} \cdot y_{He}}{A_{He}} = \frac{X \cdot 1 \cdot 1}{1} + \frac{Y \cdot 2 \cdot 1}{4} = X + \frac{1-X}{2}$$

$$\boxed{= \frac{1+X}{2}}$$

$$\mu^{-1} = \frac{1+3X}{4} + \frac{1+X}{2} \boxed{= \frac{3+5X}{4}}$$

$$X = 0.7 \rightarrow \mu = 0.62$$

Mean Molecular Weight

$$\boxed{\mu_I^{-1} = \sum_i \frac{X_i}{A_i}}$$

$$\boxed{\mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}}$$

$$\boxed{\mu^{-1} = \mu_I^{-1} + \mu_e^{-1}}$$

- In the core, all *H* is converted to *He* (X=0)

$$\boxed{X = 0 \rightarrow \mu = 1.3}$$

- Adding metals (fully ionized)

$$\mu_I^{-1} = \frac{X}{A_H} + \frac{Y}{A_{He}} + \frac{Z}{A_Z} \quad = X + \frac{Y}{4} + \frac{Z}{A_Z}$$

$$\mu_e^{-1} = \frac{X \cdot Z_H \cdot y_H}{A_H} + \frac{Y \cdot Z_{He} \cdot y_{He}}{A_{He}} + \frac{Z \cdot \frac{A_Z}{2} \cdot y_Z}{A_Z} \quad = X + \frac{Y}{2} + \frac{Z}{2}$$

$$\mu^{-1} = 2X + \frac{3}{4}Y + \underbrace{\cancel{\frac{Z}{A_Z}}}_{0} + \frac{Z}{2} \quad \boxed{= 2X + \frac{3}{4}Y + \frac{1}{2}Z}$$

$$\boxed{\begin{array}{l} X = 1 \rightarrow \mu = 0.5 \\ Z = 1 \rightarrow \mu = 2 \end{array}}$$

Equation of State

Equations of state may be derived assuming Local Thermodynamic Equilibrium (LTE)

At any position in the star, thermodynamic equilibrium holds locally even though it does not hold globally.

Particle-particle and photon-particle mean free paths are short relative to other length/time scales.

For example, the pressure scale height (height over which the pressure changed by a factor of *e*) is much longer than the mean free path.

$$\lambda_P = -\left(\frac{d\ln P}{dr}\right)^{-1} = -\left(\frac{1}{P}\frac{dP}{dr}\right)^{-1} = \frac{P}{g\rho} \quad \sim R \quad \text{in most of star}$$

Compare that to the mean free path $\lambda_\gamma \sim 1\text{cm}$

Distribution Functions

Statistical mechanics gives us the phase-space density of a particle species.

$$\boxed{n(p) = \frac{1}{h^3}\sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j + \varepsilon(p) - \mu\right)/kT\right] \pm 1}}$$

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3$$

- $p = |\vec{p}|$: momentum
- g_j : degeneracy of state *j* (# of states having same energy)
- ε_j : energy of state *j* relative to some reference level
- $\varepsilon(p)$: kinetic energy $\varepsilon(p) = \left(p^2c^2 + m^2c^4\right)^{1/2} - mc^2$

NR: $\boxed{pc \ll mc^2}$ $= \left(1 + \frac{p^2c^2}{m^2c^4}\right)^{1/2} mc^2 - mc^2 \approx \left(1 + \frac{1}{2}\frac{p^2c^2}{m^2c^4}\right) mc^2 - mc^2 = \frac{p^2}{2m}$

UR: $\boxed{pc \gg mc^2}$ $= \left(p^2c^2 + m^2c^4\right)^{1/2} - mc^2 \approx pc$

- $\mu = \left(\frac{\partial E}{\partial N}\right)$: chemical potential $\sum_i \mu_i dN_i = 0$ e.g., $H^+ + e^- \rightarrow H^0 + \gamma$

$$1\mu_{H^+} + 1\mu_{e^-} - 1\mu_{H^0} = 0$$

- $\pm$: + for fermions (Fermi-Dirac), - for bosons (Bose-Einstein)

Distribution Functions

We can get equation of state quantities from *n(p)*

Integrate over all phase space assuming spherical symmetry

- Space density

$$n = \int_0^\infty n(p)\, 4\pi p^2\, dp$$

- Internal energy
  (per unit volume)

$$E = \int_0^\infty \varepsilon(p)\, n(p)\, 4\pi p^2\, dp$$

- Pressure

$$P = \frac{1}{3}\int_0^\infty (p \cdot v)\, n(p)\, 4\pi p^2\, dp$$

Kinetic theory

$$P \cdot V = \frac{1}{3}\sum_i m_i v_i^2 \rightarrow P = \frac{1}{3V}\sum_i p_i v_i$$

Blackbody Radiation

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j + \varepsilon(p) - \mu\right)/kT\right] \pm 1}$$

For photons:

- $g = 2$ (2 polarization states)
- $\varepsilon_\gamma = 0$ (no excited states)
- $\varepsilon(p) = pc$ (fully relativistic)
- $\mu_\gamma = 0$
- bosons, so "-"

$$n(p) = \frac{1}{h^3} \frac{2}{\exp(pc/kT) - 1}$$

Blackbody Radiation

Density

$$\boxed{n(p) = \frac{1}{h^3}\frac{2}{\exp(pc/kT) - 1}}$$

$$n_\gamma = \int_0^\infty n(p) 4\pi p^2\, dp \qquad = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 dp}{\exp(pc/kT) - 1}$$

$$= \frac{8\pi}{h^3}\left(\frac{kT}{c}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} \quad \approx 2.4$$

$$= 2.4\frac{8\pi}{h^3}\left(\frac{kT}{c}\right)^3 \qquad \boxed{= 20.28 T^3 \mathrm{cm}^{-3}}$$

$$\frac{pc}{kT} = x$$

$$dp = \frac{kT}{c} dx$$

$$p^2 = \left(\frac{kT}{c}\right)^2 x^2$$

Blackbody Radiation

Energy

$$\boxed{n(p) = \frac{1}{h^3}\frac{2}{\exp(pc/kT)-1}}$$

$$\frac{pc}{kT} = x$$

$$dp = \frac{kT}{c}dx$$

$$p^3 = \left(\frac{kT}{c}\right)^3 x^3$$

$$E_\gamma = \int_0^\infty (pc)n(p)4\pi p^2\,dp = \frac{8\pi c}{h^3}\int_0^\infty \frac{p^3 dp}{\exp(pc/kT)-1}$$

$$= \frac{8\pi c}{h^3}\left(\frac{kT}{c}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x-1} = \frac{\pi^4}{15}$$

$$= \frac{\pi^4}{15}\frac{8\pi c}{h^3}\left(\frac{kT}{c}\right)^4 = \left(\frac{8\pi^5 k^4}{15h^3c^3}\right)T^4 \quad \boxed{= aT^4}$$

$$a = 7.5\times 10^{-15}\ \mathrm{erg\ cm^{-3}K^{-4}}$$

Blackbody Radiation

Pressure

$$\boxed{n(p) = \frac{1}{h^3}\frac{2}{\exp(pc/kT) - 1}}$$

$$P_\gamma = \frac{1}{3}\int_0^\infty (pc)\,n(p)\,4\pi p^2\,dp \quad = \frac{1}{3}E_\gamma \quad \boxed{= \frac{1}{3}aT^4}$$

$$P = (\gamma - 1)E \quad = \frac{1}{3}E \quad \rightarrow \gamma = \frac{4}{3}$$

Blackbody Radiation

Spectrum

$$\boxed{n(p) = \frac{1}{h^3}\frac{2}{\exp(pc/kT)-1}}$$

The energy density of photons with momentum between *p* and *p*+*dp* is

$$\varepsilon(p)n(p)4\pi p^2 dp \qquad = \frac{8\pi}{h^3}\frac{(pc)p^2 dp}{\exp(pc/kT)-1}$$

The energy density of photons with frequency between *v* and *v*+*dv* is

$$\left(\begin{array}{c} pc = h\nu \\ dp = (h/c)d\nu \end{array}\right) \qquad \frac{8\pi}{h^3}\frac{h\nu(h\nu/c)^2(h/c)d\nu}{\exp(h\nu/kT)-1}$$

$$\boxed{B_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT)-1}d\nu}$$

Planck function

$$\left(B_\nu : \mathrm{erg\ cm^{-3}\ Hz^{-1}}\right)$$

Ideal Monoatomic Gas

$$\boxed{n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j + \varepsilon(p) - \mu\right)/kT\right] \pm 1}}$$

For atoms:

- $\varepsilon_j = \varepsilon_0$ (single energy state)
- $\varepsilon(p) = \dfrac{p^2}{2m}$ (non-relativistic)
- $\mu/kT \ll -1$ (+/-1 term can be neglected)

$$n(p) = \frac{1}{h^3} \frac{g}{\exp\left[\left(\varepsilon_0 + p^2/2m - \mu\right)/kT\right]}$$

$$\boxed{n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}}$$

Ideal Monoatomic Gas

Density

$$\boxed{n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}}$$

$$n = \int_0^\infty n(p) 4\pi p^2\, dp \quad = \underbrace{\frac{4\pi g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT}}_{A} \int_0^\infty e^{-p^2/2mkT} p^2\, dp$$

$$p^2/2mkT = x$$

$$dp = \frac{mkT}{p} dx$$

$$p = (2mkT)^{1/2} x^{1/2}$$

$$= A \int_0^\infty e^{-x} (2mkTx) \frac{mkT}{(2mkTx)^{1/2}} dx$$

$$= \frac{A}{2} (2mkT)^{3/2} \int_0^\infty x^{1/2} e^{-x}\, dx \quad = \frac{\sqrt{\pi}}{2}$$

$$= A \frac{\pi^{1/2}}{4} (2mkT)^{3/2} \qquad \boxed{= \frac{g}{h^3} (2\pi mkT)^{3/2} e^{\mu/kT} e^{-\varepsilon_0/kT}}$$

Ideal Monoatomic Gas

Pressure

$$\boxed{n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}}$$

$$P = \frac{1}{3}\int_0^\infty (pv)\,n(p)\,4\pi p^2\,dp = \frac{1}{3}\underbrace{\frac{4\pi g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT}}_{A}\int_0^\infty \frac{p^2}{m} e^{-p^2/2mkT} p^2\,dp$$

$$= A\frac{1}{3m}\int_0^\infty e^{-x}(2mkTx)^2\frac{mkT}{(2mkTx)^{1/2}}\,dx$$

$$p^2/2mkT = x$$

$$dp = \frac{mkT}{p}dx$$

$$p = (2mkT)^{1/2}x^{1/2}$$

$$= \frac{1}{3m}\frac{A}{2}(2mkT)^{5/2}\int_0^\infty x^{3/2}e^{-x}\,dx$$

$$= \frac{1}{3m}\frac{A}{2}(2mkT)^{5/2}\left[\underbrace{-x^{3/2}e^{-x}\Big|_0^\infty}_{0} + \frac{3}{2}\int_0^\infty x^{1/2}e^{-x}\,dx\right]$$

$$u = x^{3/2} \qquad du = (3/2)x^{1/2}dx$$

$$v = -e^{-x} \qquad dv = e^{-x}$$

$$= kT\frac{A}{2}(2mkT)^{3/2}\int_0^\infty x^{1/2}e^{-x}\,dx \qquad \boxed{= nkT}$$

Ideal Monoatomic Gas

Energy

$$n(p) = \frac{g}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} e^{-p^2/2mkT}$$

$$E = \int_0^\infty \left(p^2/2m\right) n(p) 4\pi p^2 \, dp \qquad = \frac{3}{2} P \qquad \boxed{= \frac{3}{2} nkT}$$

$$P = (\gamma - 1) E \qquad = \frac{2}{3} E \qquad \rightarrow \gamma = \frac{5}{3}$$

Degenerate Fermions

$$\boxed{n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left[\left(\varepsilon_j + \varepsilon(p) - \mu\right)/kT\right] \pm 1}}$$

For fermions:

- $g = 2$ (2 spin states)
- $\varepsilon_0 = mc^2$ (no excited states)
- $\varepsilon(p) = \left(p^2c^2 + m^2c^4\right)^{1/2} - mc^2$
  $= mc^2\left(\sqrt{1 + (p/mc)^2} - 1\right)$
- fermions, so "+"

$$\boxed{n(p) = \frac{2}{h^3} \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right)/kT\right] + 1}}$$

Degenerate Fermions

$$\boxed{n(p) = \frac{2}{h^3} \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right)/kT\right] + 1}}$$

A completely degenerate gas behaves as if $T \to 0$

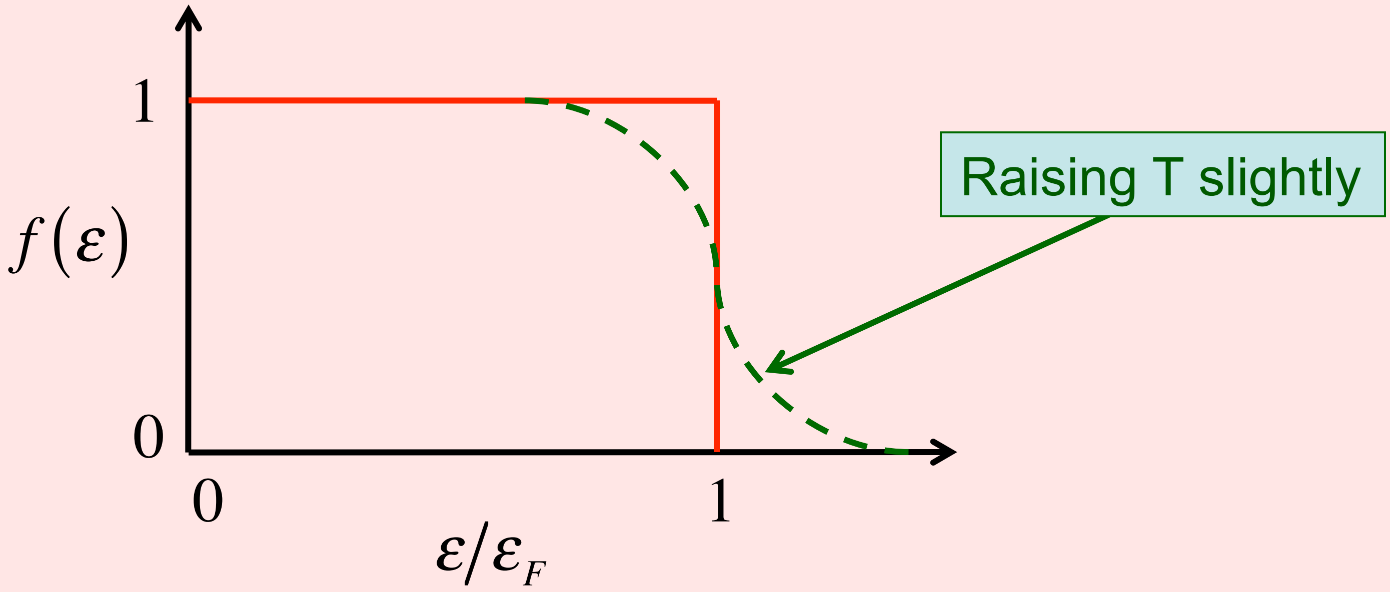
The probability that an energy state is occupied is

$$f(\varepsilon) = \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right)/kT\right] + 1} = \begin{cases} 1 \quad \text{if} \quad \varepsilon < \left(\mu - mc^2\right) \\ 0 \quad \text{if} \quad \varepsilon > \left(\mu - mc^2\right) \end{cases}$$

Critical energy: Fermi energy $\varepsilon_F = \mu - mc^2$

Particles cannot have greater energy than this.

Degenerate Fermions



Fermi momentum p_F

$$x = \frac{p}{mc} \rightarrow x_F = \frac{p_F}{mc} \qquad \varepsilon_F = mc^2\left(\sqrt{1+x_F^2} - 1\right)$$

Chemical potential is thus $\mu_F = mc^2 + \varepsilon_F$

This is the maximum total energy of particles

Degenerate Fermions

Density

$$\boxed{n(p) = \frac{2}{h^3} \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right)/kT\right] + 1}}$$

$$n = \int_0^\infty n(p) 4\pi p^2 \, dp \qquad = \frac{8\pi}{h^3} \int_0^{p_F} p^2 \, dp \qquad = \frac{8\pi}{h^3} \frac{p_F^3}{3}$$

$$\boxed{= \frac{8\pi}{3} \left(\frac{mc}{h}\right)^3 x_F^3}$$

For electrons: $n_e = \dfrac{8\pi}{3} \left(\dfrac{m_e c}{h}\right)^3 x_F^3 = 5.9 \times 10^{29} x_F^3 \ \text{cm}^{-3}$

Degenerate Fermions

Pressure

$$n(p) = \frac{2}{h^3} \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right)/kT\right] + 1}$$

$$P = \frac{1}{3}\int_0^\infty (pv)\, n(p)\, 4\pi p^2\, dp$$

$$= \frac{8\pi}{3h^3}\int_0^{p_F} v \cdot p^3\, dp$$

$$= \frac{8\pi}{3mh^3}\int_0^{p_F} \frac{p^4\, dp}{\sqrt{1 + (p/mc)^2}}$$

$$\varepsilon(p) = mc^2\left(\sqrt{1 + (p/mc)^2} - 1\right)$$

$$v = \frac{d\varepsilon}{dp}$$

$$= mc^2 \frac{1}{2}\left(1 + \left(\frac{p}{mc}\right)^2\right)^{-1/2} \cdot 2\frac{p}{mc} \cdot \frac{1}{mc}$$

$$= \frac{p}{m}\left[1 + \left(\frac{p}{mc}\right)^2\right]^{-1/2}$$

Degenerate Fermions

Pressure

$$P = \frac{8\pi}{3mh^3}\int_0^{p_F} \frac{p^4\,dp}{\sqrt{1+\left(p/mc\right)^2}}$$

$$\frac{p}{mc} = x$$

$$dp = mc\,dx$$

$$p^4 = (mc)^4 x^4$$

$$= \frac{8\pi}{3mh^3}\int_0^{x_F} \frac{(mc)^5 x^4\,dx}{\sqrt{1+x^2}} = \frac{8\pi m^4 c^5}{3h^3}\int_0^{x_F} \frac{x^4\,dx}{\sqrt{1+x^2}}$$

$$\boxed{= \frac{\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2\left[x_F\left(2x_F^2-3\right)\left(1+x_F^2\right)^{1/2} + 3\sinh^{-1} x_F\right]}$$

Degenerate Fermions

Energy

$$n(p) = \frac{2}{h^3} \frac{1}{\exp\left[\left(\varepsilon(p) - \left(\mu - mc^2\right)\right)/kT\right] + 1}$$

$$E = \int_0^\infty \varepsilon(p) n(p) 4\pi p^2 \, dp \quad = \frac{8\pi}{h^3} \int_0^{p_F} mc^2 \left(\sqrt{1 + (p/mc)^2} - 1 \right) p^2 \, dp$$

$$= \frac{8\pi}{h^3} mc^2 \int_0^{x_F} \left(\sqrt{1 + x^2} - 1 \right) (mc)^3 x^2 \, dx$$

$$= 8\pi \left(\frac{mc}{h} \right)^3 mc^2 \int_0^{x_F} \left(\sqrt{1 + x^2} - 1 \right) x^2 \, dx$$

$$\frac{p}{mc} = x$$

$$dp = mc dx$$

$$p^2 = (mc)^2 x^2$$

$$= \frac{\pi}{3} \left(\frac{mc}{h} \right)^3 mc^2 \left[8x_F^3 \left(\sqrt{1 + x_F^2} - 1 \right) - x_F \left(2x_F^2 - 3 \right) \left(1 + x_F^2 \right)^{1/2} - 3 \sinh^{-1} x_F \right]$$

Degenerate Fermions

$$n = \underbrace{\frac{8\pi}{3}\left(\frac{mc}{h}\right)^3}_{A} x_F^3 \quad \boxed{= A x_F^3}$$

$$P = \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 mc^2 \int_0^{x_F} \frac{x^4\,dx}{\sqrt{1+x^2}} \quad \boxed{= Amc^2 \int_0^{x_F} \frac{x^4\,dx}{\sqrt{1+x^2}}}$$

$$E = 8\pi\left(\frac{mc}{h}\right)^3 mc^2 \int_0^{x_F} \left(\sqrt{1+x^2}-1\right)x^2\,dx \quad \boxed{= 3Amc^2 \int_0^{x_F} \left(\sqrt{1+x^2}-1\right)x^2\,dx}$$

Degenerate Fermions

$$\boxed{n = Ax_F^3}$$

$$\boxed{P = Amc^2 \int_0^{x_F} \frac{x^4\,dx}{\sqrt{1+x^2}}}$$

$$\boxed{E = 3Amc^2 \int_0^{x_F} \left(\sqrt{1+x^2} - 1\right) x^2\,dx}$$

Look at limiting cases: NR, UR

<u>NR:</u> $x_F \ll 1$

$$P \approx Amc^2 \int_0^{x_F} x^4\,dx = Amc^2 \frac{x_F^5}{5} = Amc^2 \frac{1}{5}\left(\frac{n}{A}\right)^{5/3}$$

$$\boxed{= \frac{mc^2}{5} A^{-2/3} n^{5/3}}$$

$$E \approx 3Amc^2 \int_0^{x_F} \left(1 + \frac{1}{2}x^2 - 1\right) x^2\,dx = \frac{3}{2} Amc^2 \int_0^{x_F} x^4\,dx \quad \boxed{= \frac{3}{2}P}$$

$$P = (\gamma - 1)E = \frac{2}{3}E \quad \rightarrow \gamma = \frac{5}{3}$$

Degenerate Fermions

$$n = A x_F^3$$

$$P = Amc^2 \int_0^{x_F} \frac{x^4\,dx}{\sqrt{1+x^2}}$$

$$E = 3Amc^2 \int_0^{x_F} \left(\sqrt{1+x^2} - 1\right) x^2\,dx$$

Look at limiting cases: NR, UR

<u>UR:</u> $x_F \gg 1$

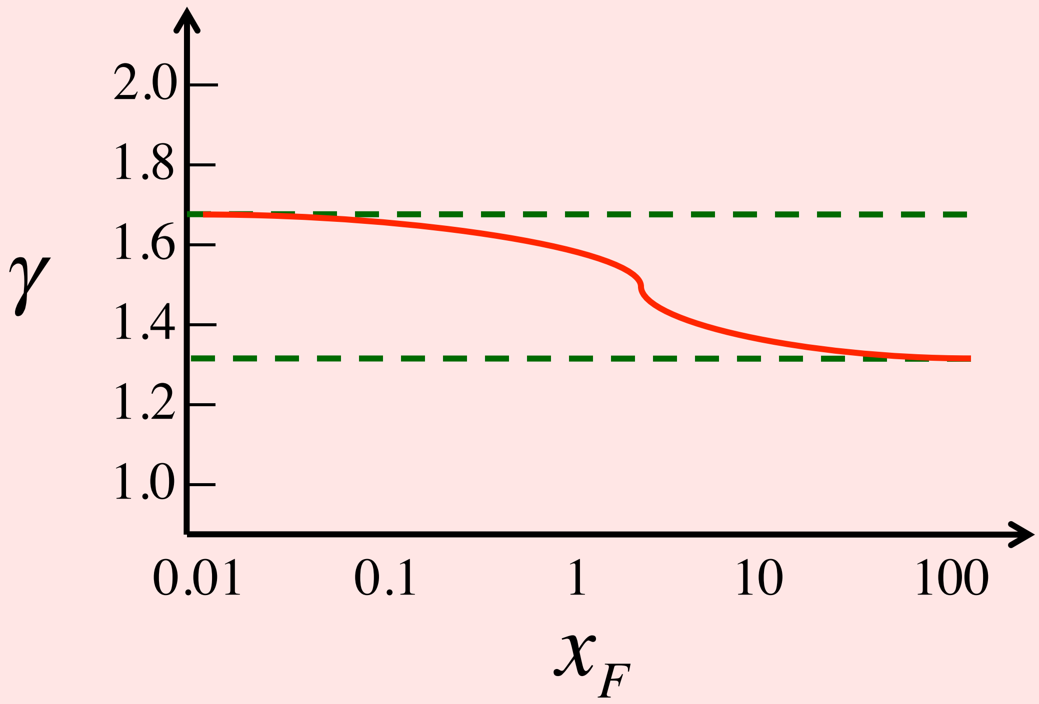
$$P \approx Amc^2 \int_0^{x_F} x^3\,dx = Amc^2 \frac{x_F^4}{4} = Amc^2 \frac{1}{4}\left(\frac{n}{A}\right)^{4/3}$$

$$\boxed{= \frac{mc^2}{4} A^{-1/3} n^{4/3}}$$

$$E \approx 3Amc^2 \int_0^{x_F} (x-1)\,x^2\,dx = 3Amc^2 \int_0^{x_F} x^3\,dx \quad \boxed{= 3P}$$

$$P = (\gamma - 1)E = \frac{1}{3}E \quad \rightarrow \gamma = \frac{4}{3}$$

Degenerate Fermions



- Completely degenerate gas has same gamma as ideal gas.
- In the relativistic limit, it behaves like radiation.

Degenerate Fermions

$$\boxed{n = \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 x_F^3}$$

Convert to mass density: $n = \frac{N_A \rho}{\mu}$

e.g., electrons: $n_e = \frac{N_A \rho}{\mu_e} = \frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3 x_F^3$

$$\frac{\rho}{\mu_e} = \frac{8\pi}{3N_A}\left(\frac{m_e c}{h}\right)^3 x_F^3 \quad \boxed{= 9.7 \times 10^5 x_F^3 \text{ g cm}^{-3}}$$

e.g., neutrons: $n_n = \frac{N_A \rho}{\mu_n} = \frac{8\pi}{3}\left(\frac{m_n c}{h}\right)^3 x_F^3$

$$\frac{\rho}{\mu_n} = \frac{8\pi}{3N_A}\left(\frac{m_n c}{h}\right)^3 x_F^3 \quad \boxed{= 6.1 \times 10^{15} x_F^3 \text{ g cm}^{-3}}$$

Degenerate Fermions

Useful approximations

- Partly degenerate gas

$$P_e \approx \left(P_{e,nd}^2 + P_{e,d}^2\right)^{1/2}$$

- Partly relativistic gas

$$P_{e,d} \approx \left(P_{e,d,NR}^{-2} + P_{e,d,UR}^{-2}\right)^{-1/2}$$

This formula picks out the smallest pressure

These interpolation formulae are good to ~2%

Equation of State Summary

- Start with $\rho,\ T,\ X_i$ at some point in star
- Assume Local Thermodynamic Equilibrium
- Total gas pressure is the sum of components

$$\boxed{P = P_{\text{rad}} + P_{\text{ion}} + P_e}$$

- The radiation pressure is

$$\boxed{P_{\text{rad}} = \frac{1}{3} a T^4} \qquad \left(a = 7.5 \times 10^{-15}\,\text{erg cm}^{-3}\text{K}^{-4}\right)$$

- The ion ideal gas pressure is

$$\boxed{P_{\text{ion}} = \frac{N_A k}{\mu_I} \rho T} \qquad \left(N_A = 6.022 \times 10^{23}\,\text{mole}^{-1}\right)$$

Equation of State Summary

- The electron pressure can be a mixture of non-degenerate and degenerate pressure

$$P_e \approx \left(P_{e,nd}^2 + P_{e,d}^2\right)^{1/2}$$

- Where the electron non-degenerate pressure is

$$P_{e,nd} = \frac{N_A k}{\mu_e} \rho T$$

- And the electron degenerate pressure can be non-relativistic or relativistic

$$P_{e,d} \approx \left(P_{e,d,NR}^{-2} + P_{e,d,UR}^{-2}\right)^{-1/2}$$

Equation of State Summary

- Where the non-relativistic, degenerate pressure is

$$P_{e,d,NR} = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e}\left(\frac{N_A}{\mu_e}\right)^{5/3} \rho^{5/3}$$

- And the ultra-relativistic, degenerate pressure is

$$P_{e,d,UR} = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4}\left(\frac{N_A}{\mu_e}\right)^{4/3} \rho^{4/3}$$

- Compute the mean molecular weights from X_i

$$\mu_I^{-1} = \sum_i \frac{X_i}{A_i} \qquad \mu_e^{-1} = \sum_i \frac{X_i Z_i y_i}{A_i}$$

- Get the ionization fractions from the Saha equation.