The Boltzmann Equation

Ideal gas :

$$n = g\left(\frac{2\pi mkT}{h^2}\right)^{3/2} e^{\mu/kT} e^{-\varepsilon_0/kT}$$

Consider two species of particles: atoms in different energy levels that can be excited or de-excited via photons.

e.g., $H_1 + \gamma \rightarrow H_2$ $\qquad 1\mu_{H_1} + 0 = 1\mu_{H_2}$

$$\frac{n_1}{n_2} = \frac{g_1\left(2\pi mkT/h^2\right)^{3/2} e^{\mu_1/kT} e^{-\varepsilon_1/kT}}{g_2\left(2\pi mkT/h^2\right)^{3/2} e^{\mu_2/kT} e^{-\varepsilon_2/kT}} = \frac{g_1 e^{-\varepsilon_1/kT}}{g_2 e^{-\varepsilon_2/kT}}$$

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{-(\varepsilon_1-\varepsilon_2)/kT}$$

The Boltzmann Equation

Total number of atoms: $$n_{\text{tot}} = \sum_i n_i$$

$$\frac{n_{\text{tot}}}{n_i} = \frac{n_1}{n_i} + \frac{n_2}{n_i} + \cdots = \frac{g_1}{g_i} e^{-\varepsilon_1/kT} e^{\varepsilon_i/kT} + \frac{g_2}{g_i} e^{-\varepsilon_2/kT} e^{\varepsilon_i/kT} + \cdots$$

$$= \frac{1}{g_i} e^{\varepsilon_i/kT} \left(g_1 e^{-\varepsilon_1/kT} + g_2 e^{-\varepsilon_2/kT} + \cdots \right) = \frac{1}{g_i e^{-\varepsilon_i/kT}} \sum_j g_j e^{-\varepsilon_j/kT}$$

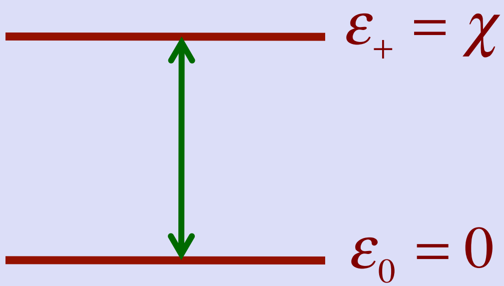
$$\frac{n_i}{n_{\text{tot}}} = \frac{g_i}{G} e^{-\varepsilon_i/kT}$$

Partition function: $$G = \sum_j g_j e^{-\varepsilon_j/kT}$$

Like an effective statistical weight for all states

The Saha Equation

Consider an atomic gas that is partly ionized



e.g., $H^+ + e^- \leftrightarrow H^0 + \gamma$ $\quad 1\mu^+ + 1\mu^- = 1\mu^0$

- charged ions in ground state $\quad n^+ = g^+ \left(\dfrac{2\pi m_p kT}{h^2} \right)^{3/2} e^{\mu^+/kT} e^{-\chi/kT}$

- free electrons $\quad n^- = g^- \left(\dfrac{2\pi m_e kT}{h^2} \right)^{3/2} e^{\mu^-/kT}$

- neutral ions in ground state $\quad n^0 = g^0 \left(\dfrac{2\pi m_p kT}{h^2} \right)^{3/2} e^{\mu^0/kT}$

Combine to eliminate chemical potentials

$$\frac{n^+ n^-}{n^0} = \frac{g^+ g^-}{g^0} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi/kT}$$

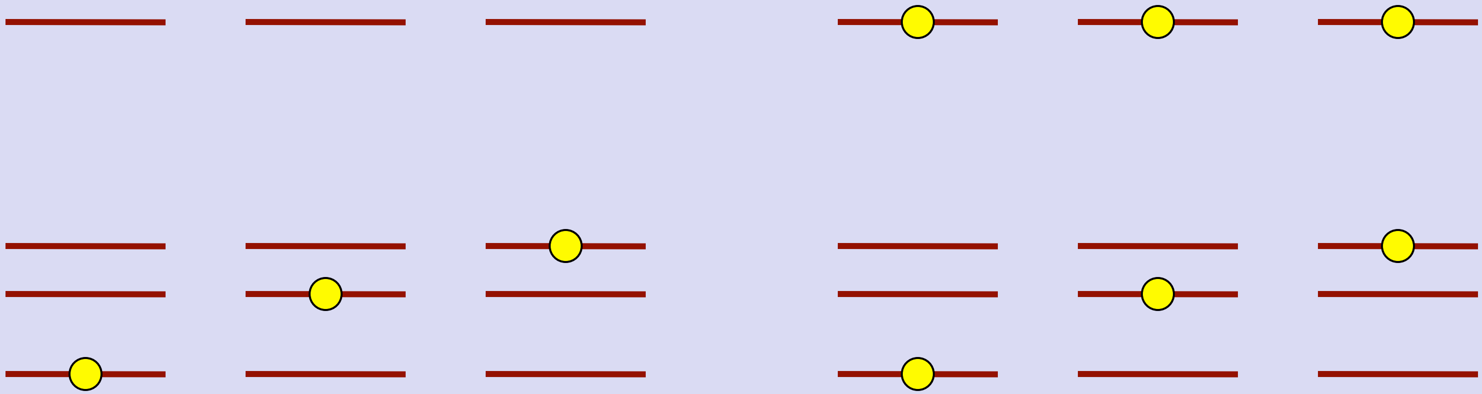
The Saha Equation

$$\frac{n^+ n^-}{n^0} = \frac{g^+ g^-}{g^0} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi/kT}$$

This is only for atoms in the ground state.

In general, we have neutral and ionized atoms in multiple states

Use Boltzmann to generalize to all states of a given atom



$$n_0^0 \qquad n_1^0 \qquad n_2^0 \qquad\qquad n_0^+ \qquad n_1^+ \qquad n_2^+$$

$$n^0 = \sum_i n_i^0 \qquad\qquad n^+ = \sum_i n_i^+$$

The Saha Equation

$$\frac{n^+ n^-}{n^0} = \frac{g^+ g^-}{g^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$

Boltzmann: $$\frac{n_i}{n_{\rm tot}} = \frac{g_i}{G} e^{-\varepsilon_i/kT}$$

$$\frac{n_0^+ n^-}{n_0^0} \qquad \frac{g_0^+ g^-}{g_0^0}$$

$$\frac{n_0^0}{n^0} = \frac{g_0^0}{G^0} \qquad \frac{n_0^+}{n^+} = \frac{g_0^+}{G^+}$$

LHS:

$$\frac{n_0^+ n^-}{n_0^0} = \frac{n^+ \frac{g_0^+}{G^+} n^-}{n^0 \frac{g_0^0}{G^0}} = \frac{n^+ n^-}{n^0}\frac{G^0}{G^+}\frac{g_0^+}{g_0^0}$$

RHS:

$$= \frac{g_0^+ g^-}{g_0^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$

$g^- = 2$

$n^- = n_e$

$$\frac{n^+ n_e}{n^0} = \frac{2G^+}{G^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$

Saha!

Can generalize to: $$\frac{n^{j+1} n_e}{n^j}$$

The Saha Equation

$$\frac{n^+ n_e}{n^0} = \frac{2G^+}{G^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$

Example: pure H

$$y = \frac{n^+}{n_I}$$

$$n_I = n^+ + n^0$$

$$n_e = n^+$$

$$G^0 \approx 2$$

$$G^+ = 1$$

$$\left.\begin{aligned} n^+ &= y n_I \\ n^0 &= (1-y) n_I \\ n_e &= y n_I \end{aligned}\right\} \quad \frac{n^+ n_e}{n^0} = \frac{y n_I \cdot y n_I}{(1-y) n_I} = \frac{y^2}{1-y} n_I$$

$$\frac{y^2}{1-y} = \frac{1}{n_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT} \qquad n_I = N_A \rho$$

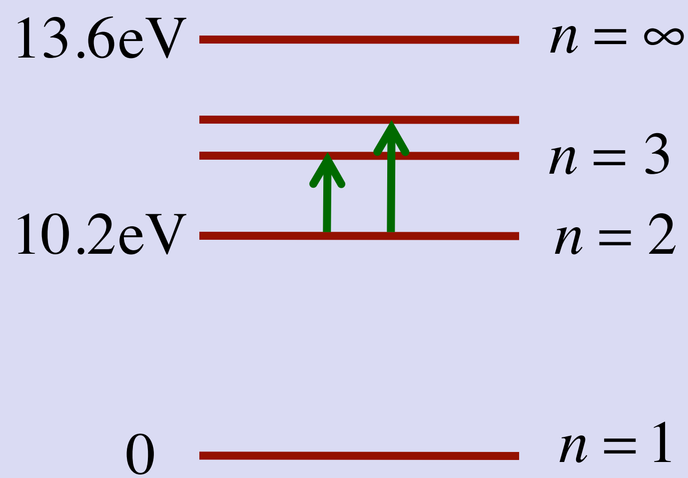
$$\frac{y^2}{1-y} = \frac{4\times 10^{-9}\,\mathrm{g\ cm^{-3}}}{\rho} T^{3/2} e^{-1.578\times 10^5/T}$$

50% ionized at T=10^4K
for low densities

The Saha Equation

Example: Balmer line strength

Strength of Balmer lines depend on fraction of H gas that is in the n=2 excited state.



We want $\dfrac{n_2^0}{n_I}$

Boltzmann gives us $\dfrac{n_2^0}{n^0} = \dfrac{g_2^0}{G^0} e^{-\varepsilon_2/kT}$

$$\left.\begin{aligned} \frac{n_2^0}{n_I} &= \frac{n_2^0}{n^0}\frac{n^0}{n_I} \\ \frac{n^0}{n_I} &= 1 - y \end{aligned}\right\}$$

$$\boxed{\frac{n_2^0}{n_I} = (1 - y)\frac{g_2^0}{G^0} e^{-\varepsilon_2/kT}}$$

$$g_2^0 = 8$$

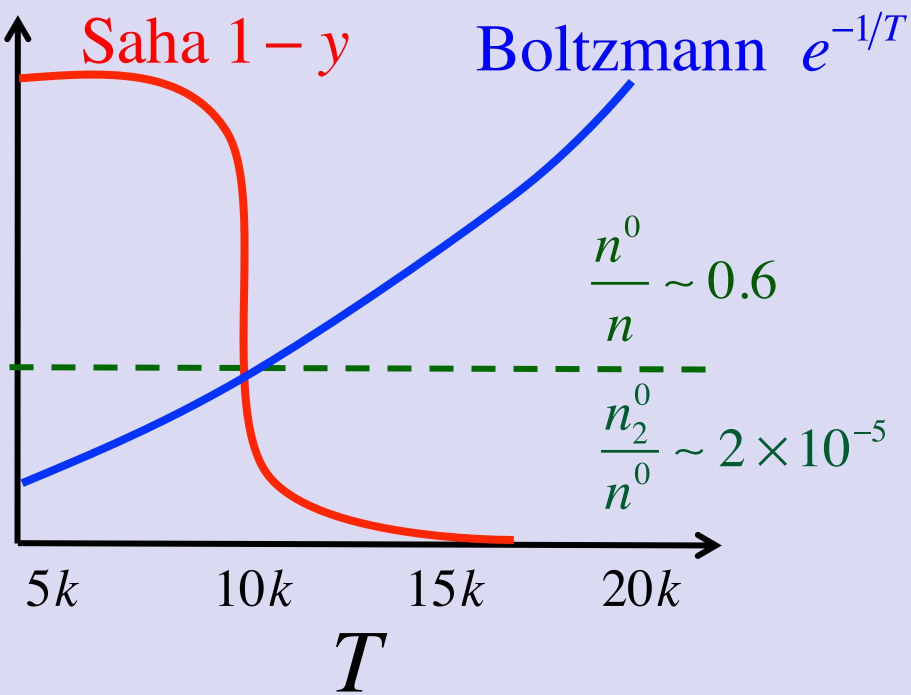
$$G^0 = \sum_i g_i e^{-\varepsilon_i/kT} \approx 2$$

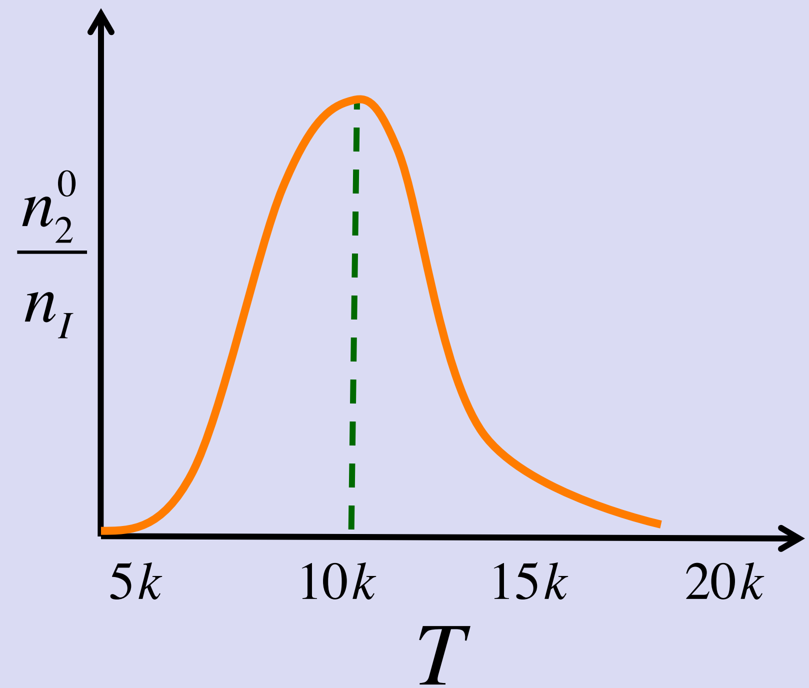
$$\varepsilon_2 = 10.2 \text{ eV}$$

The Saha Equation

$$\frac{n_2^0}{n^0} = (1 - y)\frac{g_2^0}{G^0}e^{-\varepsilon_2/kT}$$

$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$





The Saha Equation

A different way to write the Saha equation

$$\frac{n^+ n_e}{n^0} = \frac{2G^+}{G^0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi/kT}$$

$$\log\left(\frac{n^+ n_e}{n^0}\right) = \log\left(\frac{2G^+}{G^0}\right) + \frac{3}{2}\log\left(\frac{2\pi m_e kT}{h^2}\right) + \log\left(e^{-\chi/kT}\right)$$

$$\log\left(\frac{n^+}{n^0}\right) + \log n_e = \log\left(\frac{2G^+}{G^0}\right) + \frac{3}{2}\log\left(\frac{2\pi m_e k}{h^2}\right) + \frac{3}{2}\log T - \ln 10\,\frac{\chi}{kT}$$

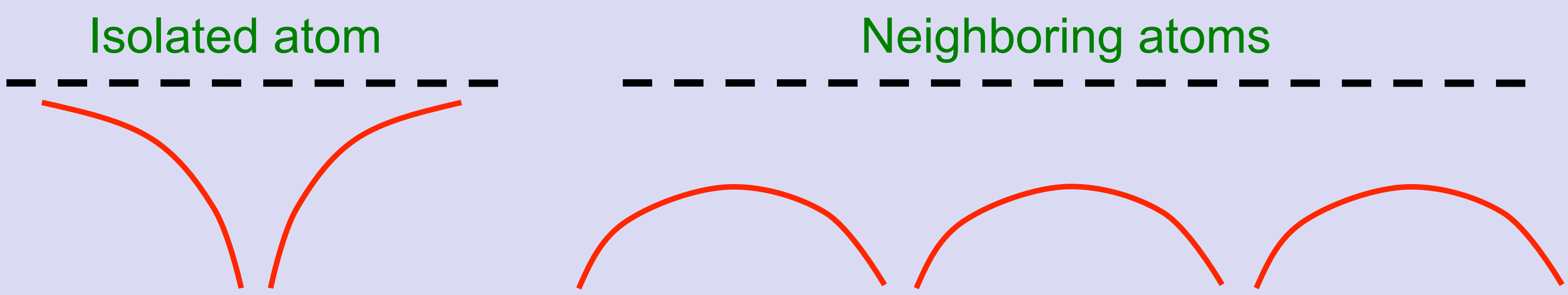
$$P_e = n_e kT \quad \rightarrow \log n_e = \log P_e - \log k - \log T$$

$$\log\left(\frac{n^+}{n^0}\right) = \log\left(\frac{2G^+}{G^0}\right) + \frac{5}{2}\log T - \log P_e - \frac{5040\chi}{T} - 0.48$$

The Saha Equation

The Saha equation works best at moderate densities

- At very low densities (e.g., corona), we must worry about whether LTE applies.
- At very high densities, atoms are not isolated.



Overlap of potentials of neighboring atoms lowers the effective ionization energy, which leads to greater ionization than Saha predicts.

For density n and separation a: $n = \left(\frac{4}{3}\pi a^3\right)^{-1}$

set a = radius of first Bohr orbit of $H = 0.5 \times 10^{-8}$ cm $\rightarrow \rho \sim 3 \text{ g cm}^{-3}$

Above this density, even the ground state is affected and all H is ionized.

The Saha Equation

- For ions of charge Z, the ratio of electrostatic energy to thermal energy is a measure of whether Coulomb effects affect the ideal-ness of an ideal gas.

$$\Gamma_C = \frac{Z^2 e^2 / a}{kT} \qquad \Gamma_C = 1 \rightarrow \rho = 85 \text{ g cm}^{-3} \left(\frac{T}{10^6 K} \right)^3$$

Important in very low mass stars

Equation of State

For pure H

- ideal gas vs. radiation

  $\rho = 1.5 \times 10^{-23} T^3$

- degenerate vs. not

  $\rho = 6 \times 10^{-9} T^{3/2}$

- neutral vs. ionized

  $\rho = 8 \times 10^{-9} T^{3/2} e^{-1.58 \times 10^5 / T}$

- pressure ionized

  $\rho \approx 1$

- Coulomb effects

  $\rho = 8.5 \times 10^{-17} T^3$

- crystalline structure

  $\rho = 4.2 \times 10^{-10} T^3$