Polytropes

First two structure equations:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

Temperature *T* does not appear in these equations explicitly, but is involved implicitly because pressure *P* usually depends on temperature via the equation of state.

In some cases, however, *P* only depends on density and, not, *T*. in these cases, the above two equations are sufficient to define a solution.

A degenerate gas is such a case.

Polytropes

Assume a "polytropic relation" holds throughout the star:

$$P = K\rho^{1+\frac{1}{n}}$$

K : constant

n : polytropic index

Polytropes are useful in two situations:

- The equation of state is really polytropic

  Completely degenerate gas

  - Non-relativistic $P \sim \rho^{5/3}$ $\rightarrow n = 3/2$
  - Ultra-relativistic $P \sim \rho^{4/3}$ $\rightarrow n = 3$

- The equation of state + an additional constraint yields a polytropic relation.
  - Isothermal ideal gas $T = T_0, \quad P \sim \rho T \sim \rho$ $\rightarrow n = \infty$
  - Fully convective star: convection maintains a fixed *T* gradient

$$T \sim P^{2/5}, \quad P \sim \rho T \sim \rho P^{2/5} \rightarrow P \sim \rho^{5/3}$$ $\rightarrow n = 3/2$

Polytropes: Lane-Emden Equation

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2} \quad \rightarrow \frac{r^2}{\rho}\frac{dP}{dr} = -Gm \quad \rightarrow \frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -G\frac{dm}{dr}$$

$$\rightarrow \frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -G4\pi r^2\rho \quad \rightarrow \frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho$$

Make this equation dimensionless. First density,

$$\rho(r) = \rho_c\theta^n(r) \qquad \rho_c = \rho(r=0)$$

$$P = K\rho^{1+\frac{1}{n}} \rightarrow P = K\rho_c^{1+\frac{1}{n}}\theta^{n+1} = P_c\theta^{n+1} \qquad \left(P_c = K\rho_c^{1+\frac{1}{n}}\right)$$

Polytropes: Lane-Emden Equation

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho \qquad \rho = \rho_c\theta^n \qquad P = P_c\theta^{n+1}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho_c\theta^n}\frac{d}{dr}\left(P_c\theta^{n+1}\right)\right] = -4\pi G\rho_c\theta^n$$

$$\rightarrow \frac{P_c}{\rho_c}\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\theta^n}(n+1)\theta^n\frac{d\theta}{dr}\right] = -4\pi G\rho_c\theta^n$$

$$\rightarrow \frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta}{dr}\right] = -\theta^n$$

Next, make radius dimensionless: $\xi = \frac{r}{r_n}$ $\qquad r_n = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}$

Polytropes: Lane-Emden Equation

$$r_n^2 \frac{1}{(\xi r_n)^2} \frac{d}{d(\xi r_n)} \left[(\xi r_n)^2 \frac{d\theta}{d(\xi r_n)} \right] = -\theta^n$$

$$\rightarrow \boxed{\frac{1}{\xi^2} \frac{d}{d\xi} \left[\xi^2 \frac{d\theta}{d\xi} \right] = -\theta^n}$$

Lane-Emden equation

$$\rightarrow \frac{1}{\xi^2} \left(2\xi \frac{d\theta}{d\xi} + \xi^2 \frac{d^2\theta}{d\xi^2} \right) = -\theta^n$$

$$\rightarrow \frac{d^2\theta}{d\xi^2} + \frac{2}{\xi} \frac{d\theta}{d\xi} + \theta^n = 0$$

$$\boxed{\theta'' = -\frac{2}{\xi}\theta' - \theta^n}$$

Polytropes: Lane-Emden Equation

$$\theta'' = -\frac{2}{\xi}\theta' - \theta^n$$

$$\xi = \frac{r}{r_n}$$

$$\rho(r) = \rho_c \theta^n(r)$$

Boundary conditions

- Center

$r = 0 \rightarrow \xi = 0$ $\quad$ $\rho = \rho_c \rightarrow \theta = 1$ $\quad$ $\frac{d\rho}{dr} = 0 \rightarrow \theta' = 0$

- Surface is at first zero crossing of $\theta(\xi)$

$\rho = 0 \rightarrow \theta = 0$ $\quad$ $\xi = \xi_1$

Solving Lane-Emden

The Lane-Emden equation can be integrated numerically outward until $\theta = 0$.

- Start at $\xi = 0$. In steps of $d\xi$ compute θ'', θ', θ. Stop when $\theta = 0$.
- Get value of ξ_1 , as well as full $\theta(\xi)$ profile.

Polytropes: Lane-Emden Equation

- Analytic solutions exist for three cases: n=0, 1, and 5

$$n = 0: \quad \theta(\xi) = 1 - \frac{\xi^2}{6} \qquad \xi_1 = \sqrt{6}$$

$$n = 1: \quad \theta(\xi) = \frac{\sin\xi}{\xi} \qquad \xi_1 = \pi$$

$$n = 5: \quad \theta(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \qquad \xi_1 = \infty$$

- Polytropes with n>5 have infinite (divergent) mass.

- Only models with n=3/2 and n=3 are physically relevant.

$$n = \frac{3}{2}: \quad P = K\rho^{5/3} \qquad n = 3: \quad P = K\rho^{4/3}$$

Polytropes: Lane-Emden Equation

Solving a differential equation numerically

Simplest case: $\frac{dy}{dx} = f(x,y)$ Boundary condition: $y(0) = y_0$

$$y'(0) = y'_0$$

- Choose step Δx

- Start with boundary condition and evaluate y at next step

$$y(\Delta x) = y(0) + y'(0) \cdot \Delta x$$

- Repeat for all subsequent steps

$$y(x) = y(x - \Delta x) + \frac{dy}{dx}(x - \Delta x) \cdot \Delta x$$

Polytropes: Lane-Emden Equation

Solving a differential equation numerically

- Loop over steps:

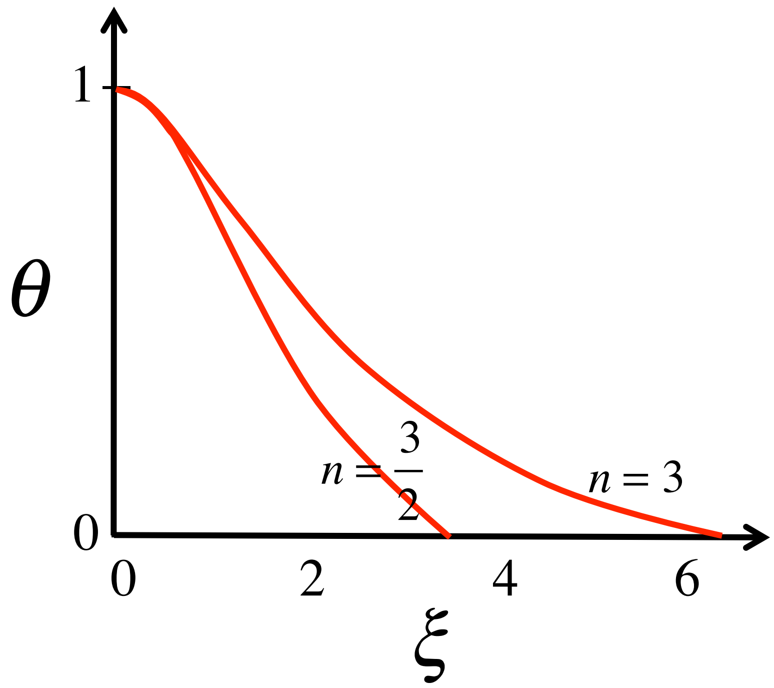
$$y[i] = y[i-1] + \frac{dy}{dx}[i-1] \cdot \Delta x$$

$$\frac{dy}{dx}[i] = \frac{dy}{dx}[i-1] + \frac{d^2y}{dx^2}[i-1] \cdot \Delta x$$

$$\frac{d^2y}{dx^2}[i] = f(x, y, y')$$

Polytropes: Lane-Emden Equation

- As *n* increases, solutions become less centrally concentrated



Polytropes

- Total mass enclosed by *r*

$$M(<r) = \int_0^r \rho(r) 4\pi r^2 \, dr$$

$$M(<\xi) = \int_0^r \rho_c \theta^n 4\pi (\xi r_n)^2 \, d(\xi r_n) \quad = 4\pi r_n^3 \rho_c \int_0^r \theta^n \xi^2 \, d\xi$$

$$= 4\pi r_n^3 \rho_c \int_0^r \xi^2 \left[-\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) \right] d\xi \qquad = 4\pi r_n^3 \rho_c \left[-\xi^2 \frac{d\theta}{d\xi} \right]$$

- Total mass in star

$$M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1) \right]$$

- Total radius of star

$$R = r_n \xi_1 = \left[\frac{(n+1) P_c}{4\pi G \rho_c^2} \right]^{1/2} \xi_1$$

- Pressure - density

$$P_c = K \rho_c^{1+\frac{1}{n}}$$

Polytropes

We have four equations → we can get 4 unknowns.

e.g., if we know the equation of state: *K*, *n*, and the mass *M*, we can

- solve Lane-Emden to get ξ_1 and $\theta'(\xi_1)$

$$\theta'' = -\frac{2}{\xi}\theta' - \theta^n$$

- use four equations to get r_n, R, ρ_c, P_c

$$P_c = K\rho_c^{1+\frac{1}{n}}$$

$$R = r_n\xi_1 = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}\xi_1$$

$$M = 4\pi r_n^3\rho_c\left[-\xi_1^2\theta'(\xi_1)\right]$$

Polytropes: Structure of a White Dwarf

$$P_c = K\rho_c^{1+\frac{1}{n}}$$

$$M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1) \right]$$

$$R = r_n \xi_1 = \left[\frac{(n+1) P_c}{4\pi G \rho_c^2} \right]^{1/2} \xi_1$$

Consider a White Dwarf composed of a degenerate, non-relativistic electron gas.

$$n = \frac{3}{2}, \quad P = K\rho^{5/3}$$

$$R^2 \sim \frac{P_c}{\rho_c^2} \sim \frac{\rho_c^{5/3}}{\rho_c^2} = \rho_c^{-1/3} \rightarrow \rho_c \sim R^{-6}$$

$$M \sim r_n^3 \rho_c \sim R^3 R^{-6} \sim R^{-3} \rightarrow R \sim M^{-1/3}$$

The radius of a White Dwarf shrinks as its mass increases!

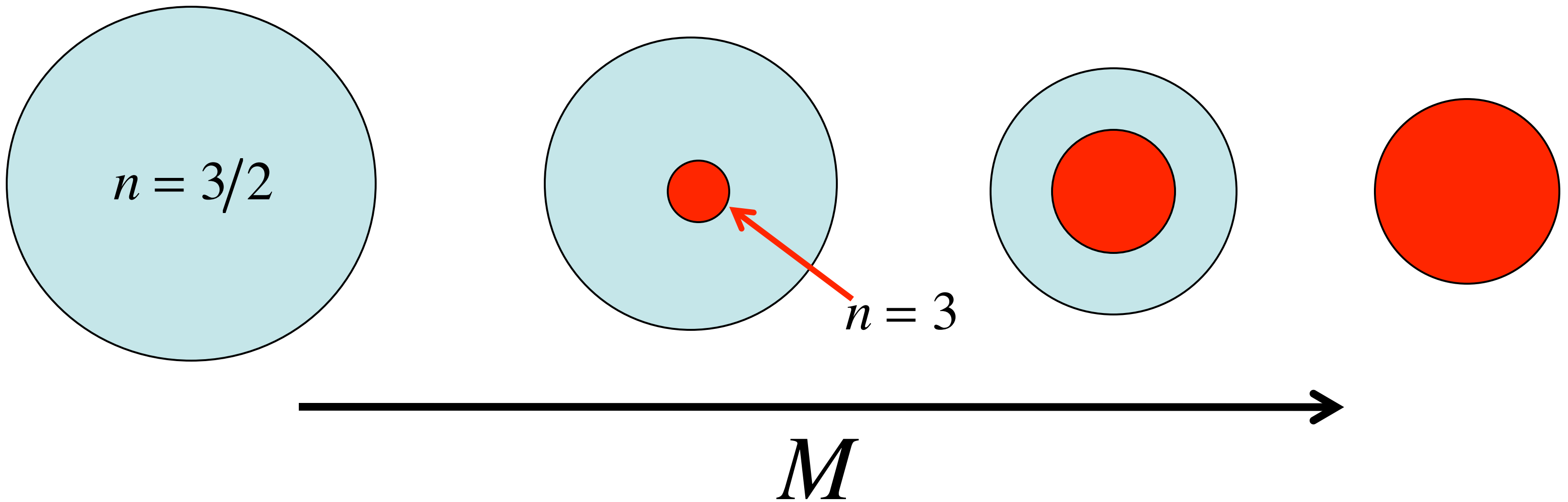
Polytropes: Structure of a White Dwarf

$$R \sim M^{-1/3} \qquad \rho_c \sim R^{-6}$$

As the white dwarf mass increases, its radius shrinks and its central density increases.

Eventually, the core will become relativistic.

As the mass keeps growing, the relativistic core grows too.



Polytropes: Structure of a White Dwarf

$$P_c = K\rho_c^{1+\frac{1}{n}}$$

$$M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right]$$

$$R = r_n \xi_1 = \left[\frac{(n+1)P_c}{4\pi G \rho_c^2}\right]^{1/2} \xi_1$$

Consider a White Dwarf composed of a degenerate, ultra-relativistic electron gas.

$$n = 3, \quad P = K\rho^{4/3}$$

$$R^2 \sim \frac{P_c}{\rho_c^2} \sim \frac{\rho_c^{4/3}}{\rho_c^2} = \rho_c^{-2/3} \rightarrow \rho_c \sim R^{-3}$$

$$M \sim r_n^3 \rho_c \sim R^3 R^{-3} \rightarrow \boxed{M = \text{const}}$$

When the white dwarf is fully relativistic, its mass decouples from its radius and central density!

Polytropes: Structure of a White Dwarf

$$P_c = K\rho_c^{1+\frac{1}{n}}$$

$$M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right]$$

$$R = r_n \xi_1 = \left[\frac{(n+1)P_c}{4\pi G \rho_c^2}\right]^{1/2} \xi_1$$

$$r_n = \left(\frac{4P_c}{4\pi G \rho_c^2}\right)^{1/2} = \left(\frac{K\rho_c^{4/3}}{\pi G \rho_c^2}\right)^{1/2} = \left(\frac{K}{\pi G}\rho_c^{-2/3}\right)^{1/2} = \left(\frac{K}{\pi G}\right)^{1/2} \rho_c^{-1/3}$$

$$M = 4\pi r_n^3 \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right] = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} \rho_c^{-1} \rho_c \left[-\xi_1^2 \theta'(\xi_1)\right]$$

$$M = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} \left[-\xi_1^2 \theta'(\xi_1)\right]$$

Plugging in numbers:

$$M = 1.46 M_\odot$$

This is the Chandrasekhar limiting mass for white dwarfs.
No white dwarfs are observed with mass greater than this.