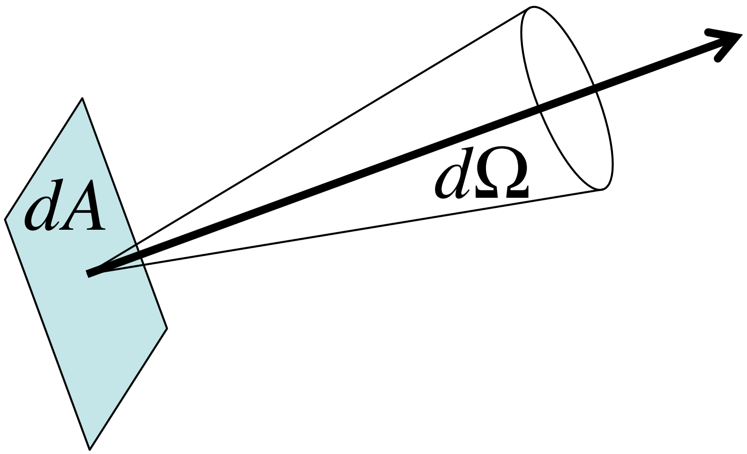
Radiative Transfer

Consider energy in the form of light, passing through a medium. The amount of energy passing through depends on location, direction of flow, time, frequency of light, area through which light is passing.



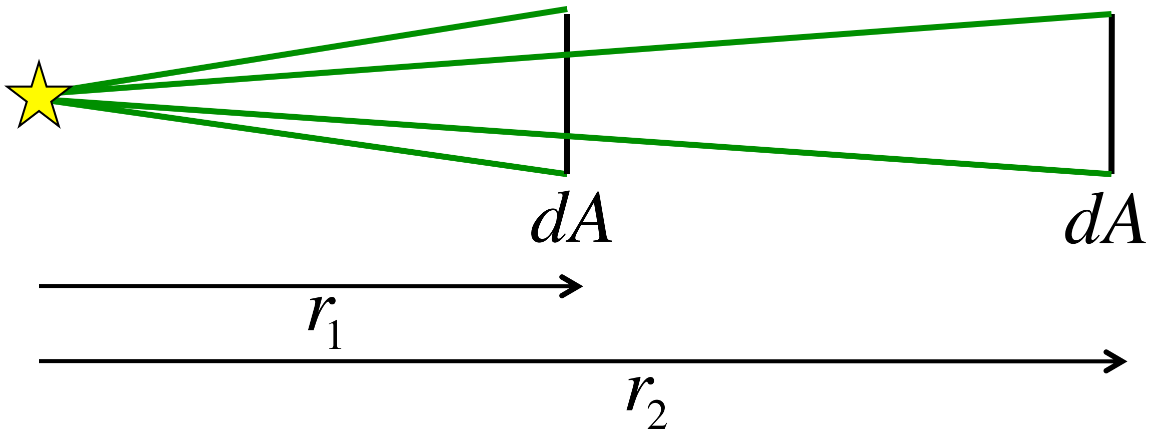
Radiative Transfer

E	erg	Energy
E_ν	$\mathrm{erg\ Hz^{-1}}$	Specific Energy
L	$\mathrm{erg\ s^{-1}}$	Luminosity
L_ν	$\mathrm{erg\ s^{-1} Hz^{-1}}$	Specific Luminosity
F	$\mathrm{erg\ s^{-1} cm^{-2}}$	Flux
F_ν	$\mathrm{erg\ s^{-1} cm^{-2} Hz^{-1}}$	Specific Flux
I	$\mathrm{erg\ s^{-1} cm^{-2} st^{-1}}$	Intensity
I_ν	$\mathrm{erg\ s^{-1} cm^{-2} st^{-1} Hz^{-1}}$	Specific Intensity

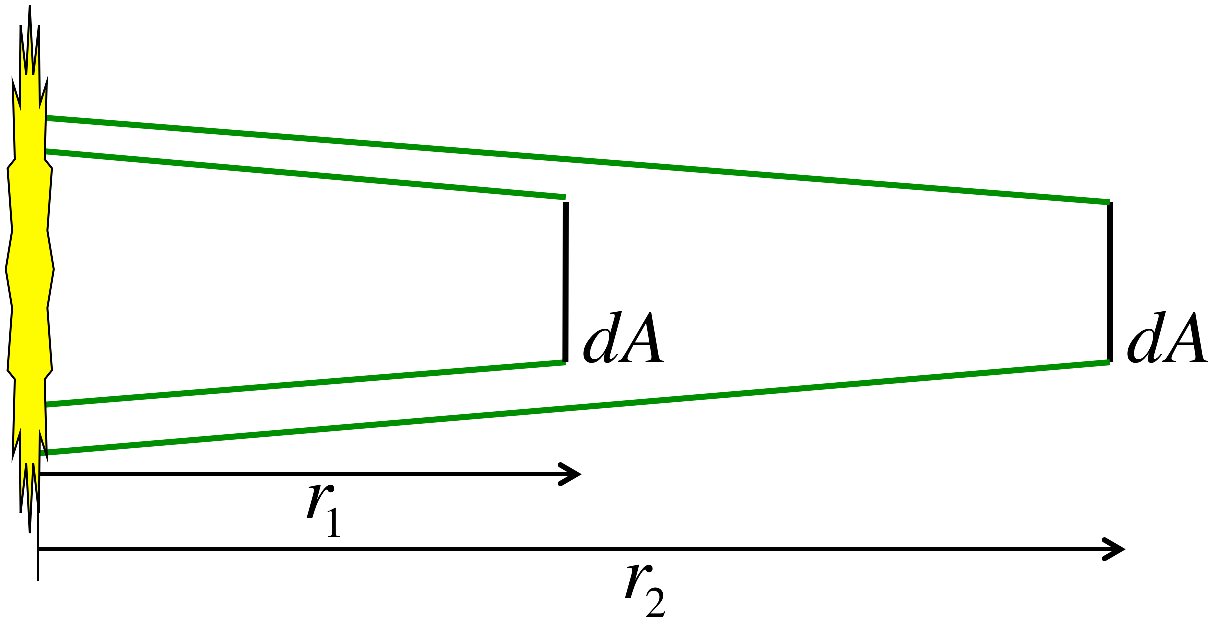
$$\boxed{dE = I_\nu \cdot dt \cdot dA \cdot d\Omega \cdot d\nu}$$

Radiative Transfer

I_ν is constant along a ray travelling on a path, whereas flux decreases as r^{-2}.



Fewer rays from a given source cross an area dA that is farther away because that area is a smaller fraction of the whole sphere at that radius.



However, an equal number of rays cross the area dA per solid angle because as dA is further away, a fixed solid angle corresponds to a larger area of emitting region by $\sim r^2$.

Radiative Transfer: Emission

Electrons can recombine with atoms or drop to lower energy levels and emit photons

<u>Emission coefficient</u>

$j \quad \mathrm{erg\ s^{-1}cm^{-2}st^{-1}cm^{-1}}$

$j_\nu \quad \mathrm{erg\ s^{-1}cm^{-2}st^{-1}Hz^{-1}cm^{-1}}$

$$dE = j_\nu \cdot dt \cdot dV \cdot d\Omega \cdot d\nu$$

$$\boxed{dI_\nu = j_\nu ds}$$

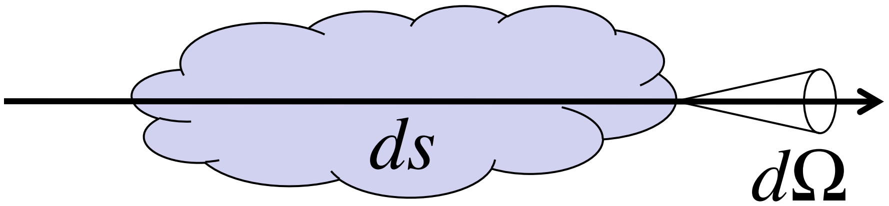
j_ν is the change of I_ν per unit length along a path.

We are only considering spontaneous emission here. Stimulated emission depends on I_ν and is more conveniently treated as a "negative absorption".

Radiative Transfer: Absorption

Absorption

As light passes through a medium, it can be absorbed.



$$\boxed{dI_\nu = -\alpha_\nu I_\nu ds}$$

$\alpha_\nu ds$: fractional loss of intensity over path *ds*

α_ν : cm^{-1} positive $\alpha_\nu \rightarrow$ energy loss

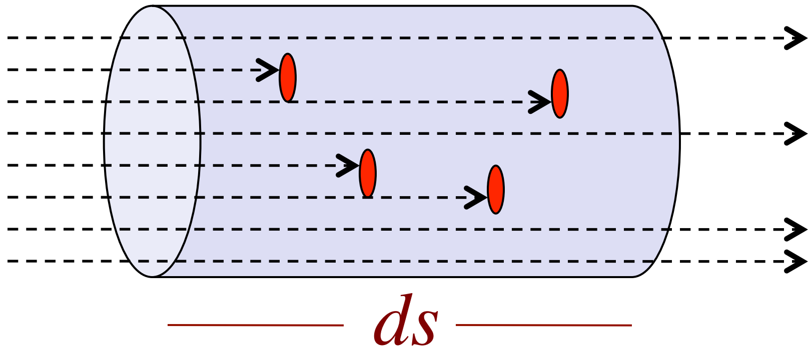
The change in intensity depends on intensity itself because the more photons try to move through a medium, the more photons can become absorbed by it.

Radiative Transfer: Absorption

If absorption is due to particles with number density n and cross-sectional area σ_ν

Consider a volume element

$$ds \cdot dA$$



Total area covered by particles:

$$n \cdot dV \cdot \sigma_\nu = n \cdot dA \cdot ds \cdot \sigma_\nu$$

Fraction of area that is covered by particles:

$$\frac{n \cdot dA \cdot ds \cdot \sigma_\nu}{dA} = n \cdot \sigma_\nu \cdot ds = \alpha_\nu ds \quad \rightarrow \quad \boxed{\alpha_\nu = n\sigma_\nu}$$

Define opacity as: $\boxed{\alpha_\nu = \rho\kappa_\nu}$ $\qquad \kappa_\nu : \mathrm{cm}^2\mathrm{g}^{-1}$

Opacity is a property of the intervening material, like a cross-sectional area per gram of material.

Radiative Transfer: Optical Depth

Define optical depth τ_ν

$$\boxed{d\tau_\nu = \alpha_\nu ds}$$

$$dI_\nu = -\alpha_\nu I_\nu ds \quad \rightarrow \frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \quad \rightarrow I_\nu = I_\nu(0)e^{-\alpha_\nu s}$$

$$dI_\nu = -I_\nu d\tau_\nu \quad \rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu \quad \rightarrow I_\nu = I_\nu(0)e^{-\tau_\nu}$$

Optical depth is the number of e-foldings change in intensity.
It is an alternative variable for path.

- Fixed ds = fixed distance
- Fixed $d\tau$ = fixed change of I_ν

Radiative Transfer Equation

Change in intensity over a path length is the sum of emission + absorption (source + sink terms).

$$dI_\nu = -\alpha_\nu I_\nu ds + j_\nu ds \quad \rightarrow \quad \boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu}$$

$$dI_\nu = -I_\nu d\tau_\nu + j_\nu \frac{d\tau_\nu}{\alpha_\nu} \quad \rightarrow \quad \boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu}$$

$$\boxed{S_\nu = \frac{j_\nu}{\alpha_\nu}}$$ Source function

In stars, the source function is close to the blackbody Planck function.

Radiative Transfer Equation

$$\frac{dI_\nu}{d\tau_\nu} + I_\nu = S_\nu \qquad \rightarrow \; e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

$$\rightarrow \frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right) = e^{\tau_\nu} S_\nu \qquad \rightarrow \; I_\nu e^{\tau_\nu} = \int_0^{\tau_\nu} e^{\tau'_\nu} S_\nu d\tau'_\nu \; + \; \mathrm{const}$$

When $\tau_\nu = 0 \;\rightarrow\; I_\nu = I_\nu(0) \;\rightarrow\; \mathrm{const} = I_\nu(0)$

$$\rightarrow \; I_\nu e^{\tau_\nu} = I_\nu(0) + \int_0^{\tau_\nu} e^{\tau'_\nu} S_\nu d\tau'_\nu$$

$$\rightarrow \; I_\nu = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu d\tau'_\nu$$

Radiative Transfer Equation

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu d\tau'_\nu$$

Final intensity is initial intensity diminished by absorption
+ integrated source function also diminished by absorption.

For $S_\nu = \text{const:}$ $I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu$

$$= I_\nu(0)e^{-\tau_\nu} + S_\nu \left[e^{-(\tau_\nu - \tau'_\nu)} \right]_0^{\tau_\nu}$$

$$\boxed{I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)}$$

as $\tau_\nu \to \infty, \quad I_\nu \to S_\nu$

Radiative Transfer: Net Flux

- If area *dA* is perpendicular to light rays

$$dF_\nu = I_\nu d\Omega$$

- If normal vector to area *dA* is at an angle θ to light rays

$$dF_\nu = I_\nu \cos\theta d\Omega$$

Flux is reduced because effective area is smaller.

- "Net flux" in the direction of $\vec{n}$ is $\boxed{F_\nu(\vec{n}) = \int I_\nu \cos\theta \, d\Omega}$

For an isotropic radiation field intensity is angle-independent.

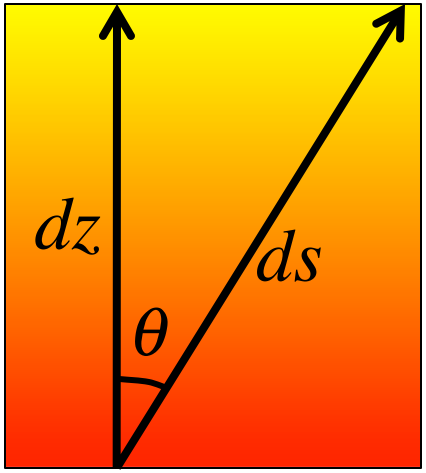
$$F_\nu(\vec{n}) = I_\nu \int \cos\theta \, d\Omega = 0$$

Radiative Transfer: Radiative Diffusion

Within a star there is a net radial flux (outward) causing a departure from isotropy. We can relate this flux to the temperature gradient.

- plane-parallel approximation
- angular dependence through θ only
- $\mu = \cos\theta$

$$\boxed{ds = dz/\mu}$$



$$\frac{dI_\nu}{ds} = -\alpha_\nu\left(I_\nu - S_\nu\right) \quad = -\rho\kappa_\nu\left(I_\nu - S_\nu\right)$$

$$\rightarrow \mu\frac{\partial I_\nu}{\partial z} = -\rho\kappa_\nu\left(I_\nu - S_\nu\right) \qquad \rightarrow I_\nu = S_\nu - \frac{\mu}{\rho\kappa_\nu}\frac{\partial I_\nu}{\partial z}$$

Radiative Transfer: Radiative Diffusion

$$\boxed{I_\nu = S_\nu - \underbrace{\frac{\mu}{\rho\kappa_\nu}\frac{\partial I_\nu}{\partial z}}}$$

$\rho\kappa_\nu$ is the fractional loss of I_ν per unit length.

$(\rho\kappa_\nu)^{-1}$ is the mean-free-path.

This second term is the change in intensity over the mean-free-path.

This is small $\rightarrow I_\nu \approx S_\nu \approx B_\nu$

To first order: $I_\nu \approx B_\nu - \dfrac{\mu}{\rho\kappa_\nu}\dfrac{\partial B_\nu}{\partial z}$

Integrate to get flux in z-direction: $F_\nu(z) = \int I_\nu \cos\theta \, d\Omega \; = -2\pi \int_{+1}^{-1} I_\nu(\mu,z)\mu \, d\mu$

$$d\Omega = 2\pi \sin\theta d\theta = -2\pi d\mu$$

Radiative Transfer: Radiative Diffusion

$$\boxed{I_\nu \approx B_\nu - \frac{\mu}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}}$$

$$F_\nu(z) = 2\pi \int_{-1}^{+1} I_\nu(\mu,z)\,\mu\,d\mu$$

B_ν is isotropic so only the derivative term has a non-zero angle dependence.

$$F_\nu(z) = 2\pi \int_{-1}^{+1} \left(-\frac{\mu}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z} \right) \mu\,d\mu \quad = -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\int_{-1}^{+1} \mu^2\,d\mu$$

$$= -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\left[\frac{\mu^3}{3}\right]_{-1}^{+1} \quad = -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\frac{2}{3}$$

$$\boxed{F_\nu(z) = -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z}}$$

Radiative Transfer: Radiative Diffusion

$$\boxed{F_\nu(z) = -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z}}$$

Total flux (integrating over frequency):

$$F(z) = \int_0^\infty F_\nu(z)\,d\nu \qquad = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu$$

Define the Rosseland mean opacity:

$$\boxed{\frac{1}{\rho\kappa_R} \equiv \frac{\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu}}$$

Can be computed for any material, at any density and any temperature (albeit with difficulty).

Radiative Transfer: Radiative Diffusion

$$F(z) = -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}d\nu$$

$$= -\frac{4\pi}{3}\frac{\partial T}{\partial z}\frac{1}{\rho\kappa_R}\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu$$

$$= -\frac{4\pi}{3}\frac{\partial T}{\partial z}\frac{1}{\rho\kappa_R}\frac{\partial}{\partial T}\int_0^\infty B_\nu\, d\nu$$

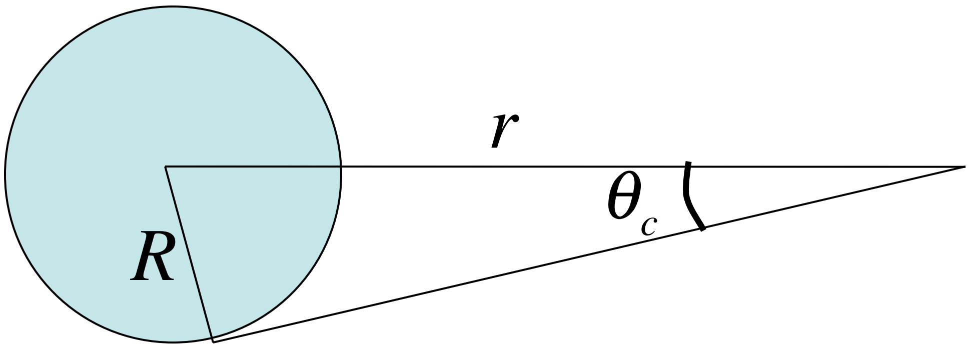
$$= -\frac{4\pi}{3\rho\kappa_R}\frac{\partial T}{\partial z}\frac{\partial B}{\partial T}$$

where *B* is the total intensity

Radiative Transfer: Radiative Diffusion

Sphere of uniform brightness.
At an exterior point,

$$I = \begin{cases} B, & \theta < \theta_c \\ 0, & \theta > \theta_c \end{cases}$$



Find flux: $$F = \int I \cos\theta \, d\Omega \quad = B \int_0^{2\pi} d\phi \int_0^{\theta_c} \cos\theta \sin\theta \, d\theta$$

$$= 2\pi B \int_0^{\theta_c} \cos\theta \sin\theta \, d\theta \quad = 2\pi B \int_0^{\sin\theta_c} x \, dx \quad = 2\pi B \frac{\sin^2\theta_c}{2}$$

$$\sin\theta_c = \frac{R}{r}$$

$$\boxed{F = \pi B \left(\frac{R}{r}\right)^2}$$

At surface: $r=R$ $$\boxed{F = \pi B}$$

Radiative Transfer: Radiative Diffusion

We found that the flux coming from an enclosed sphere is:

Also, for a blackbody:

$$\left.\begin{aligned} F &= \pi B \\ F &= \sigma T^4 \end{aligned}\right\} B = \frac{\sigma T^4}{\pi}$$

$$\boxed{F(z) = -\frac{4\pi}{3\rho\kappa_R}\frac{\partial T}{\partial z}\frac{\partial B}{\partial T}}$$

$$\frac{\partial B}{\partial T} = \frac{4\sigma T^3}{\pi}$$

$$\boxed{F(z) = -\frac{16\sigma T^3}{3\rho\kappa_R}\frac{\partial T}{\partial z}}$$

<u>Equation of radiative diffusion</u>

- Applies provided that quantities change slowly on scale of mean free path.
- Material properties enter through κ_R

Radiative Transfer: Radiative Diffusion

Spherical symmetry: $z \to r$

$$\left.\begin{aligned} L_r &= F(r)4\pi r^2 \\ \sigma &= \frac{ac}{4} \end{aligned}\right\} L_r = -\frac{16\left(\frac{ac}{4}\right)T^3}{3\rho\kappa_R}\frac{dT}{dr}4\pi r^2$$

$$\boxed{L_r = -\frac{16\pi acr^2T^3}{3\rho\kappa_R}\frac{dT}{dr}}$$

$$\boxed{\frac{dT}{dr} = -\frac{3\rho\kappa_R L_r}{16\pi acr^2T^3}}$$

The Eddington Limit

Photons carry momentum → absorption of photons must lead to a force.

Spherical symmetric source with luminosity *L*

- Energy flux at distance *r* : $\dfrac{L}{4\pi r^2}$ $\left(\mathrm{erg\ s^{-1}cm^{-2}}\right)$

- Momentum flux : $\dfrac{L}{4\pi c r^2}$ $\left(\mathrm{erg\ cm^{-3}}\right)$

  $\left(E = pc\right)$

- Multiply by opacity to get force per unit mass

$$\left(F = \frac{dp}{dt}\right) \qquad F_{\mathrm{rad}} = \frac{\kappa L}{4\pi c r^2} \quad \left(\mathrm{erg\ cm\ g^{-1}}\right)$$

Opacity is the fraction of momentum flux absorbed per unit mass.

The Eddington Limit

- Inward force per mass due to gravity: $$F_{\text{grav}} = \frac{GM}{r^2}$$

Radiation balances gravity when $F_{\text{rad}} = F_{\text{grav}}$

$$\frac{\kappa L}{4\pi c r^2} = \frac{GM}{r^2} \quad \rightarrow \quad L = \frac{4\pi c GM}{\kappa}$$

At greater luminosities, $F_{\text{rad}} > F_{\text{grav}}$ and gas will be blown away.

- Assume opacity is due to Thompson scattering by free electrons

$$\kappa = \frac{\alpha}{\rho} = \frac{n\sigma}{\rho} = \frac{\sigma}{m} = \frac{\sigma_T}{m_H}$$

$$\boxed{L_{\text{edd}} = \frac{4\pi c G M m_H}{\sigma_T}}$$

$$\boxed{L_{\text{edd}} = 3.2 \times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot}$$

The Eddington Limit

$$\boxed{L_{\mathrm{edd}} = 3.2 \times 10^{4} \left(\frac{M}{M_{\odot}} \right) L_{\odot}}$$

This is the Eddington limit

Assumptions:

- Thompson scattering only

  other opacity sources increase opacity → lower L_{edd}
- spherical symmetry

- Mass-Luminosity relation for very massive stars

$$\left(\frac{L}{L_{\odot}} \right) = 34.2 \left(\frac{M}{M_{\odot}} \right)^{2.4} = L_{\mathrm{edd}} \rightarrow M_{\max} \sim 100 M_{\odot}$$

Formation of more massive stars cannot be spherically symmetric

Opacity Sources: Electron Scattering

The opacity of a material depends on the composition (X, Y, Z) the temperature T and the density ρ of gas.

$$\kappa = \kappa_0 \rho^n T^{-s}$$

1. <u>Electron scattering</u>

$$\kappa = \frac{n\sigma}{\rho} \qquad = \frac{n_e \sigma_e}{\rho}$$

In an ionized mixture of *H* and *He*: $n_e = \frac{\rho N_A}{\mu_e}$ $\qquad \mu_e = \frac{2}{1+X}$

$$\kappa_e = \frac{\rho N_A (1+X)}{2} \frac{\sigma_e}{\rho} \qquad = \frac{\sigma_e N_A (1+X)}{2}$$

Opacity Sources: Electron Scattering

If electrons are non-degenerate and non-relativistic, their cross-section is equal to the Thompson cross-section.

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \qquad = 0.6652 \times 10^{-24}\,\mathrm{cm}^2$$

$$\kappa_e = \frac{\sigma_T N_A (1+X)}{2}$$

$$\boxed{\kappa_e = 0.2(1+X)\,\mathrm{cm}^2\mathrm{g}^{-1}}$$

Cannot use if
- heavy elements are abundant or gas is partially ionized
- density is high → degenerate
- temperature is high → relativistic

e^- opacity has no frequency, density or temperature dependence

$$\boxed{\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = s = 0}$$

Opacity Sources: Free-Free Absorption

2. <u>Free-Free Absorption</u> A free electron cannot absorb a photon because energy and momentum cannot both be conserved. However, the presence of a charged ion near the electron can make this possible.

$$\gamma + e^- + \text{ion} \rightarrow e^- + \text{ion}$$

inverse of Bremsstrahlung

$$\kappa_{ff} \approx 10^{23} \frac{\rho}{\mu_e} \frac{Z_c^2}{\mu_I} T^{-3.5} \text{cm}^2 \text{g}^{-1}$$

Z_c : average nuclear charge

Since free electrons are required, free-free opacity will be negligible for $T < 10^4\,\text{K}$ since *H* will not be ionized.

$$\boxed{\kappa_{ff} \approx 4 \times 10^{22} (X+Y)(1+X) \rho T^{-3.5} \text{cm}^2 \text{g}^{-1}}$$

- fully ionized
- no metals

$$\boxed{\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = 1, s = 3.5}$$

Kramers opacity

Opacity Sources: Bound Absorption

3. Bound-Free Absorption

A photon gets absorbed by a bound electron, ionizing it.

$$\kappa_{bf} \approx 4\times10^{25} Z(1+X)\rho T^{-3.5}\,\mathrm{cm^2 g^{-1}}$$

• $T > 10^4$ K

$$\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = 1, s = 3.5$$

Kramers opacity

4. Bound-Bound Absorption

A photon gets absorbed and causes a transition between bound energy levels in an atom.

• Very complex calculation: absorption line profiles, line broadening.
• ~10 times smaller than f-f or b-f

Kramers opacity

Opacity Sources: H^- Absorption

5. <u>H^- Opacity</u>

At low temperature, an extra electron can attach to the H atom.

H^- has an ionization potential of 0.75 eV so it's very easy to ionize if T > a few thousand K.

$$\kappa_{H^-} \approx 2.5 \times 10^{-31} \left(\frac{Z}{0.02} \right) \rho^{1/2} T^{9} \mathrm{cm}^2 \mathrm{g}^{-1}$$

$$3000 < T < 6000\mathrm{K}$$
$$10^{-10} < \rho < 10^{-5}\mathrm{g}$$
$$X \sim 0.7, \quad 0.001 < Z < 0.03$$

Relevant to Sun's atmosphere!

$$\kappa = \kappa_0 \rho^n T^{-s}, \qquad n = 0.5, s = -9$$

Opacity Sources: Tabulated Opacities

In practice, stellar models use opacities calculated using detailed physics.

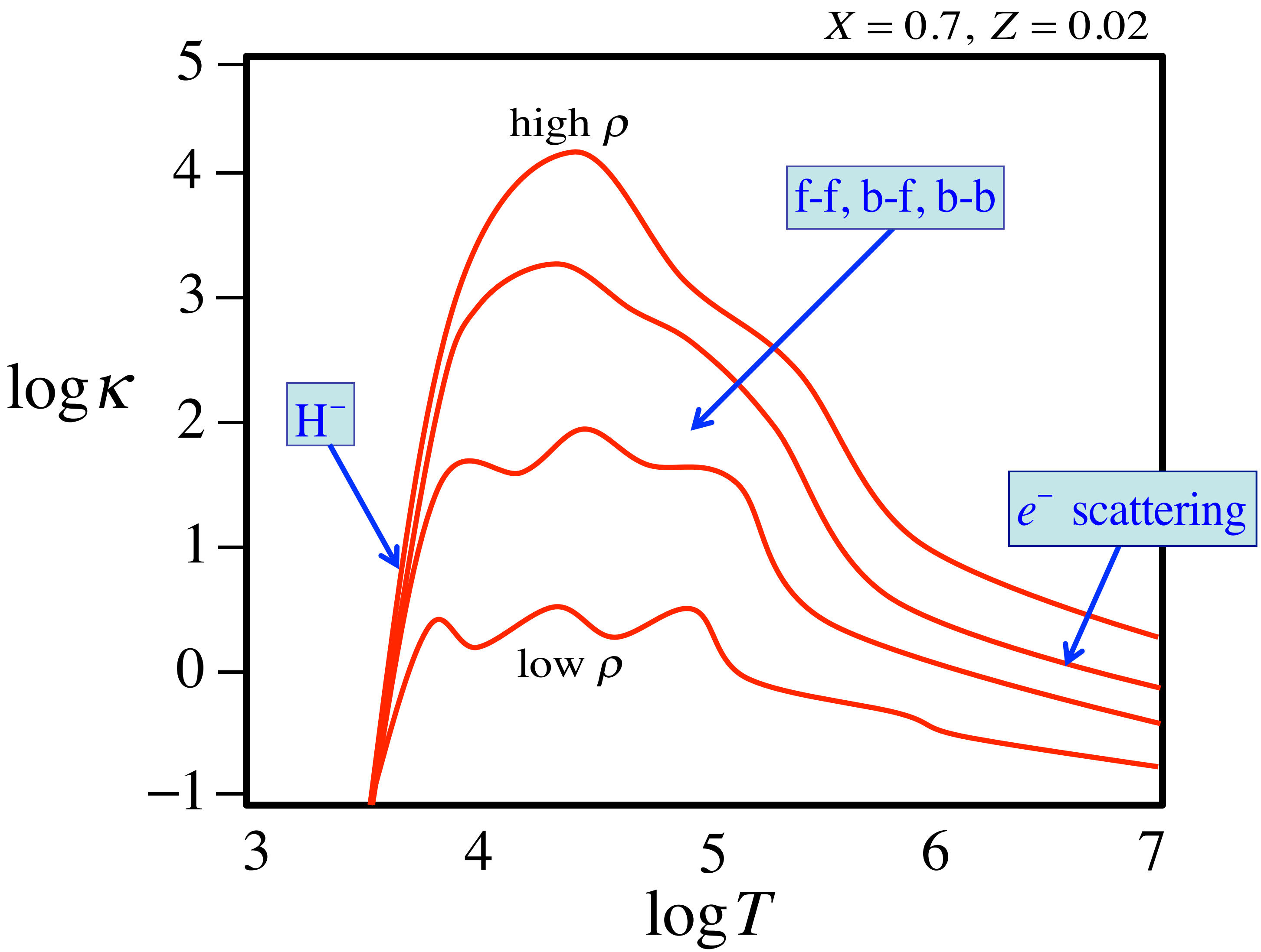
As a function of $\rho, T, X, Y, Z + \text{breakdown of metals}$

- Calculate all the relevant b-b, b-f, f-f, H^-, e^-, scattering effects.
- Get a correct Rosseland mean opacity.

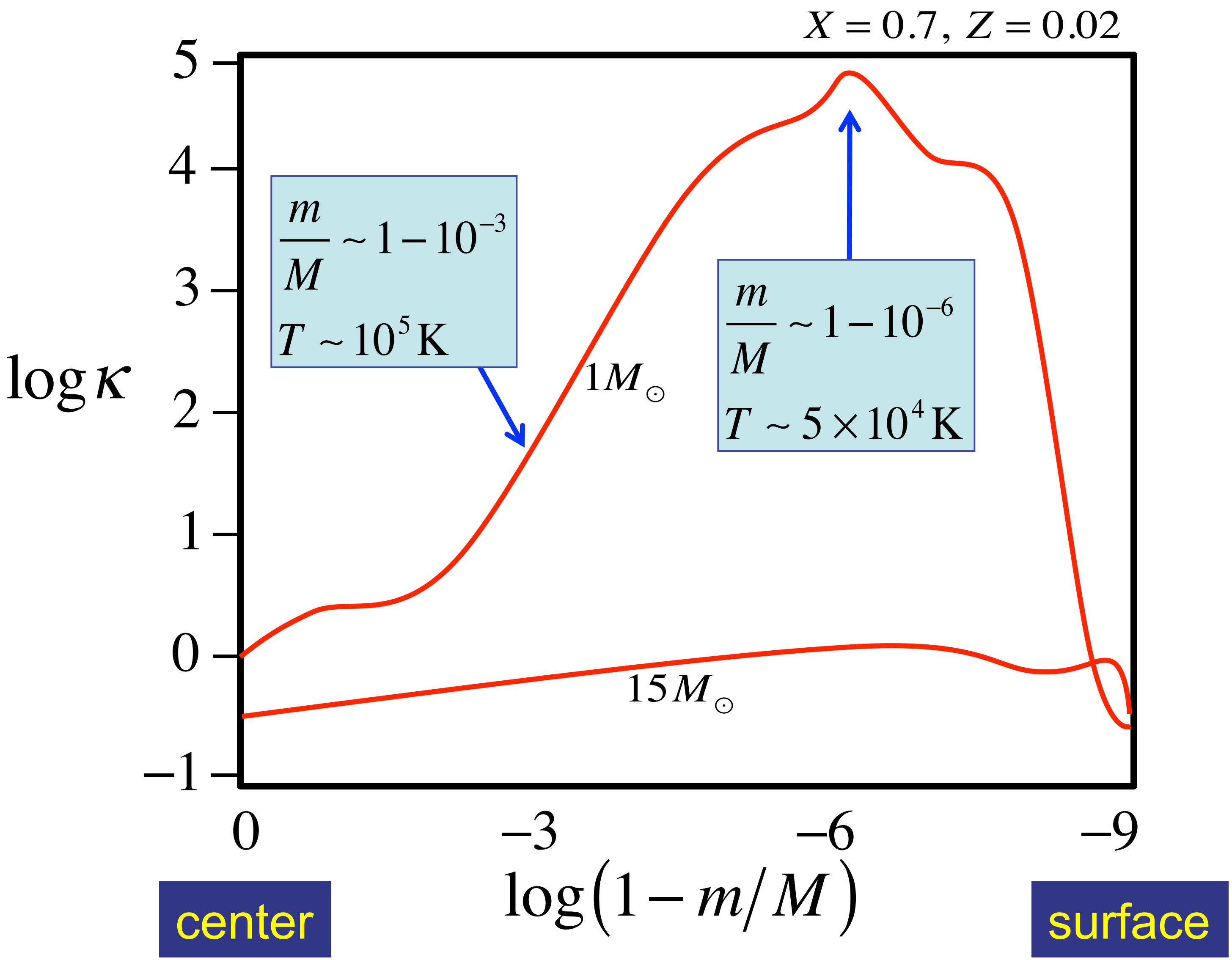
- LANL 1960s following defense calculations
- OPAL Rogers & Iglesias (1992), Iglesias & Rogers (1996)
- Opacity Project Seaton et al. (1994)

- Opacity tables do not cover whole ρ, T, X, Y, Z space. Extrapolation is dangerous.
- Need to do smart interpolation.
- Differences of as much as 30% between projects occur.

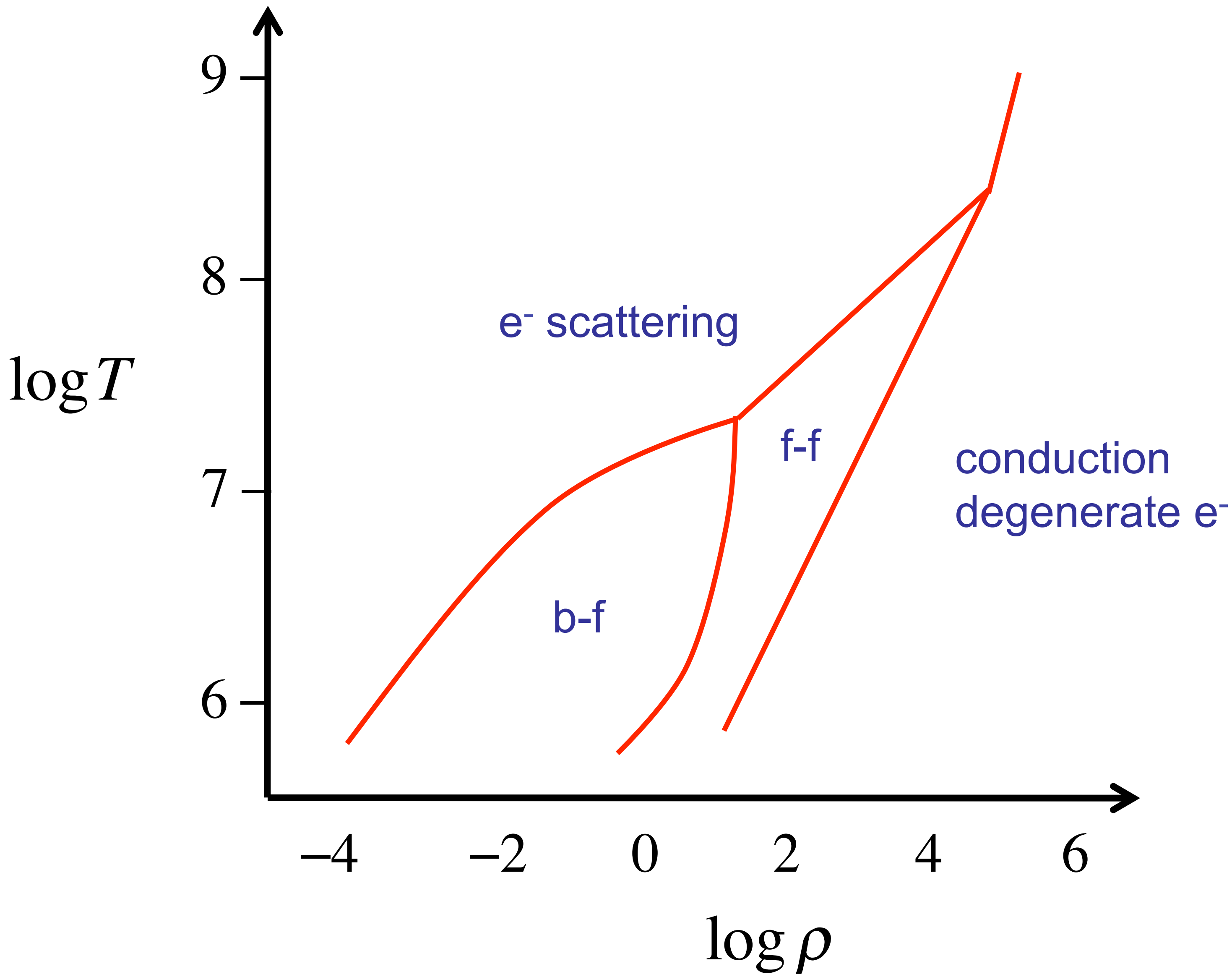
Opacity Sources: Tabulated Opacities



Opacity Sources: Tabulated Opacities



Opacity Sources: Tabulated Opacities



Heat Transfer by Conduction

When the density gets very high, degenerate electrons transfer heat by conduction in addition to providing hydrostatic support.

Total energy flux is additive: $F_{\text{tot}} = F_{\text{rad}} + F_{\text{cond}}$

$$F_{\text{rad}} = -\frac{4acT^3}{3\rho\kappa_R}\frac{dT}{dr} \qquad F_{\text{cond}} = -\frac{4acT^3}{3\rho\kappa_{\text{cond}}}\frac{dT}{dr}$$

$$\frac{1}{\kappa_{\text{tot}}} = \frac{1}{\kappa_R} + \frac{1}{\kappa_{\text{cond}}}$$

Like resistance in a parallel circuit

- In normal stars, κ_{cond} is large → conduction is negligible

  In center of sun: $\kappa_R \sim 0.2, \quad \kappa_{\text{cond}} \sim 2\times 10^9$!!!

- In degenerate dense stars, it can be smaller than κ_R

  In center of cool white dwarf: $\kappa_R \sim 0.2, \quad \kappa_{\text{cond}} \sim 5\times 10^{-5}$