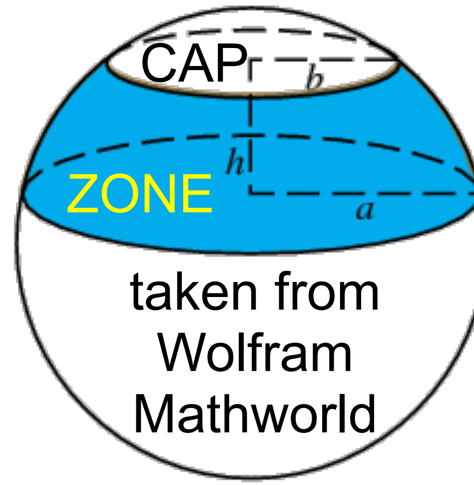
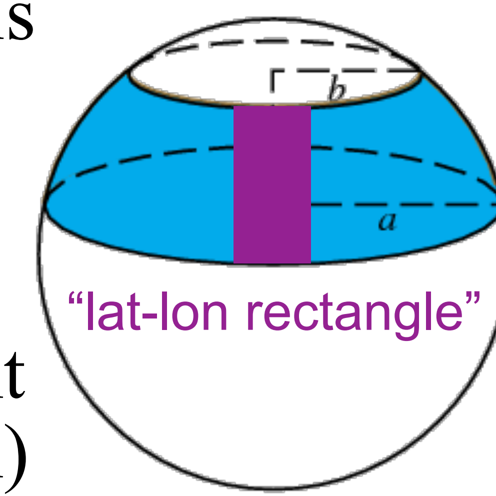
Spherical Caps

Areas on the sphere

- The area of a zone (on the unit sphere) is *$2\pi h$* in *steradians* (see the link to Wolfram MathWorld on the syllabus)



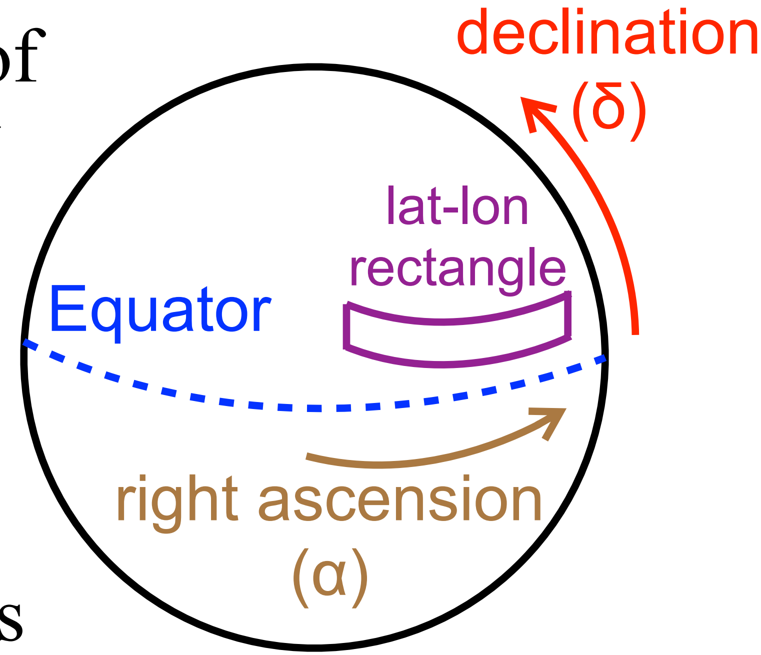
- The area of a *cap* is then *$2\pi(1-h)$*.
  – The spherical cap will come in very useful in the next lecture
- The area of a "*rectangle drawn on the sphere,*" which is a fraction of a zone, is *$f2\pi h$* where *f* is the fraction in this "*lat-lon rectangle*"

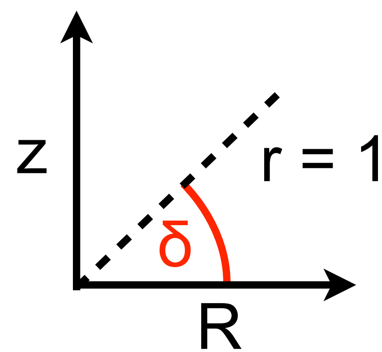


- A "*lat-lon rectangle*" as I'll call it (it doesn't have an "official" name) is bounded by lines of longitude (or Right Ascension) and latitude (or declination)

Areas on the sphere

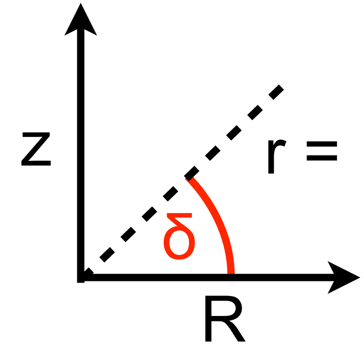
- From the coordinate discussion of a few lectures ago, we can easily find the h in $f2\pi h$
  - $h = z_2 - z_1 = \sin\delta_2 - \sin\delta_1$
- $2\pi f$ depends on the fraction of the full circle covered by the α range of interest (in radians $2\pi f$ is just the difference in α):
  - $2\pi f = (\alpha_2^{radians} - \alpha_1^{radians})$
- From $f2\pi h$, the area of a *lat-lon rectangle* bounded by α and δ is…
  - $(\alpha_2^{radians} - \alpha_1^{radians})(\sin\delta_2 - \sin\delta_1)$





$z = r\sin\delta$
$= \sin\delta$

Areas on the sphere - a note on spherical caps



$z = r\sin\delta$
$= \sin\delta$

- By definition, a spherical cap is the same area no matter where it cuts the sphere
- The radius of a cap may cover a different span in α but it is the same physical radius, and so corresponds to the same physical area
- Note the expression for the area of a spherical cap:
  - $2\pi(1\text{-}h) = 2\pi(1\text{-}z_2) = 2\pi(1\text{-}\sin\delta_2)$ in steradians
  - $= 2\pi(1\text{-}\sin\delta_2) * (180/\pi)^2$ in deg^2
- This is very close to (but not quite the same as) a cap area being $\pi\theta^2$ where θ is the cap radius drawn on the sphere
- Try, comparing, e.g., $\theta=1^{\circ}$ and $\delta_2=89^{\circ}$ to $\theta=60^{\circ}$ and $\delta_2=30^{\circ}$. From here on, we'll write $1\text{-}\sin\delta_2$ as $1\text{-}\cos\theta$

Areas on the sphere

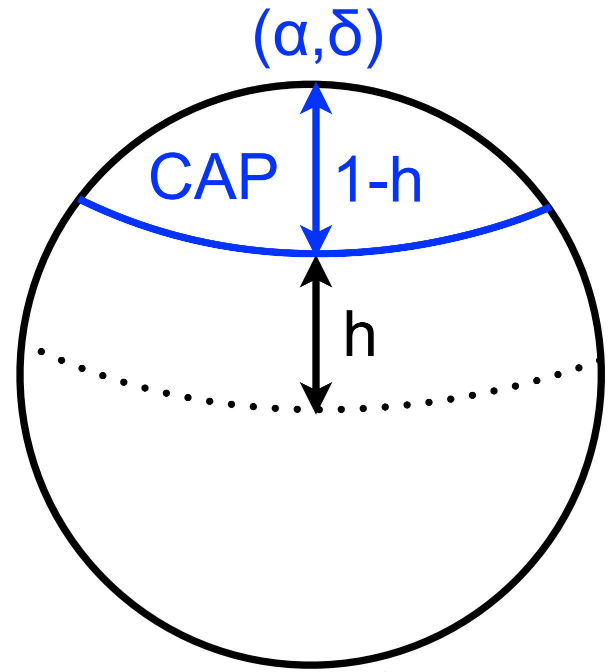
- So, *in steradians*, the area of a *lat-lon rectangle* bounded by Right Ascension α and declination δ is
  - $(\alpha_2^{radians} - \alpha_1^{radians})(sin\delta_2 - sin\delta_1)$
- Then, the area of a *lat-lon rectangle* bounded by α and δ is given by…
  - $(180/\pi)(180/\pi)(\alpha_2^{radians} - \alpha_1^{radians})(sin\delta_2 - sin\delta_1)$

…in *square degrees*

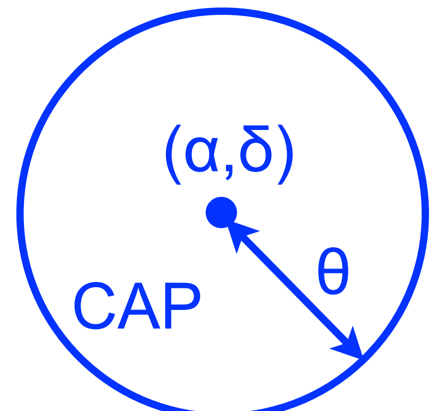
- Or, in a more compact form useful when working with astronomical coordinates (for which α is usually expressed in degrees)
  - $(180/\pi)(\alpha_2^{degrees} - \alpha_1^{degrees})(sin\delta_2 - sin\delta_1)$

The Spherical Cap

- Spherical caps are useful models for representing arbitrary regions on the surface of the sphere
- A spherical cap is centered at a specific, Right Ascension and declination (α,δ)
- The height (1-h) of a spherical cap corresponds to some radius (θ) on the surface of the unit sphere
- By intersecting *multiple* spherical caps it is possible to construct general shapes on the surface of the sphere

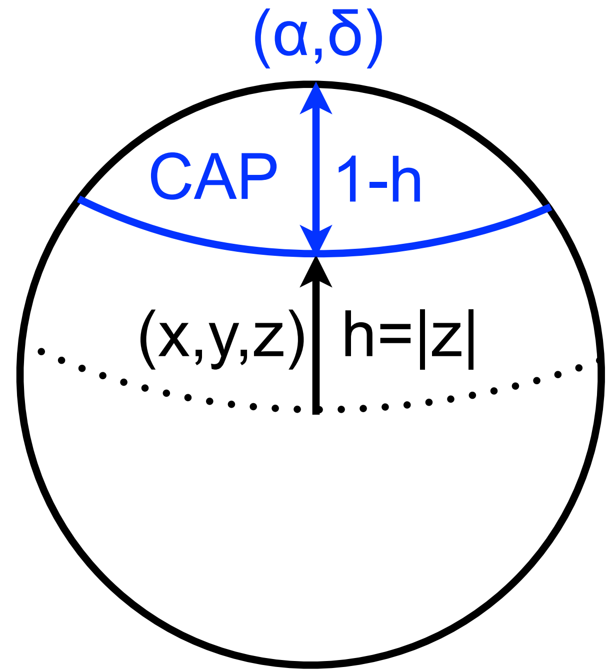


3-D view



2-D view

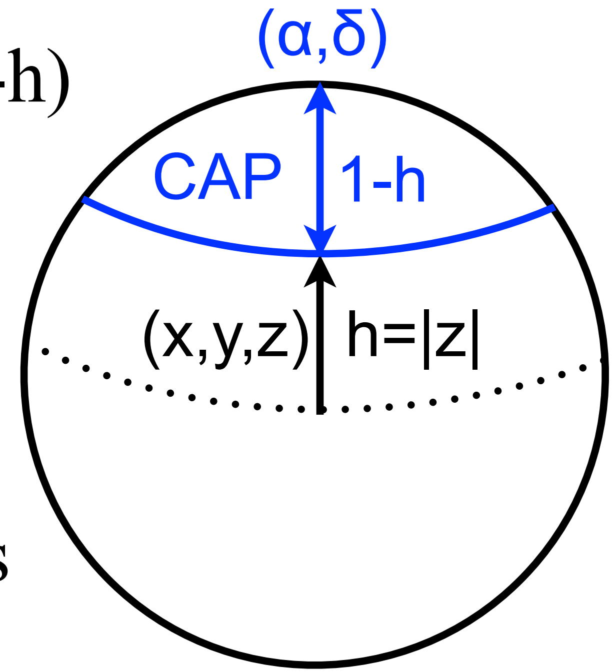
Vectorial representation of the spherical cap

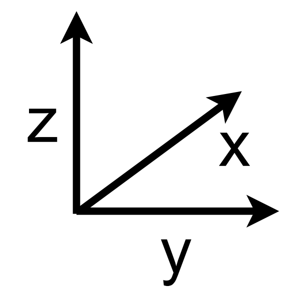


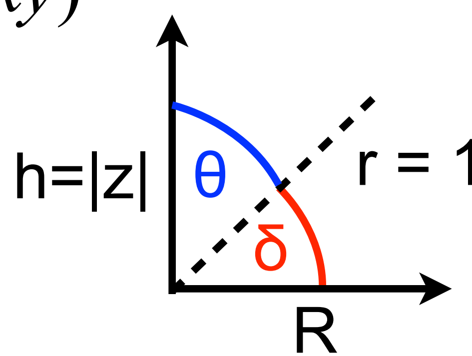
- As we know, spherical coordinates can be represented in Cartesian form (x,y,z) by the vector that points in the direction of (α,δ)
- Previously, for a *point on the surface of the sphere* we noted that $x^2 + y^2 + z^2 = 1$
- For a cap instead of a point, the *size* of the cap (which controls θ, the radius drawn on the surface of the sphere) can be controlled by $x^2 + y^2 + z^2 = h^2$
- Thus, one simple way to represent a cap is the 4-array given by x, y, z and h...we will choose the convention (x,y,z,1-h) as it is used by *Mangle* (next class' focus)

Vectorial representation of the spherical cap

- Caps can be represented by (x,y,z,1-h)
- (x,y,z) is easy to determine, it's just the Cartesian conversion from (α,δ) (e.g., using *SkyCoord*)
- For astronomers, the natural way to think about the cap size is the radius drawn on the surface of the sphere (θ from the last 2 slides, the radius we would put in *search_around_sky*)
- As shown in the diagrams to the right, 1-h = 1-cosθ
- So, (x,y,z,1-h) ≡ (x,y,z,1-cosθ)



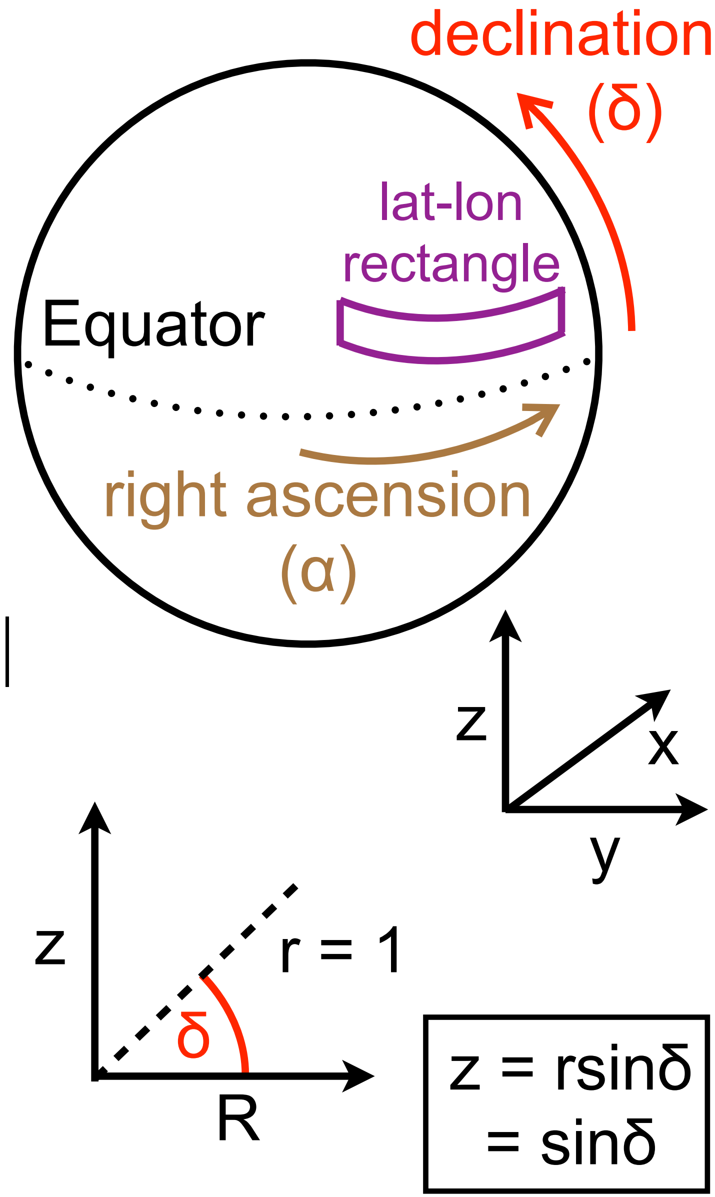




$$z = r\cos\theta$$
$$= \cos\theta$$

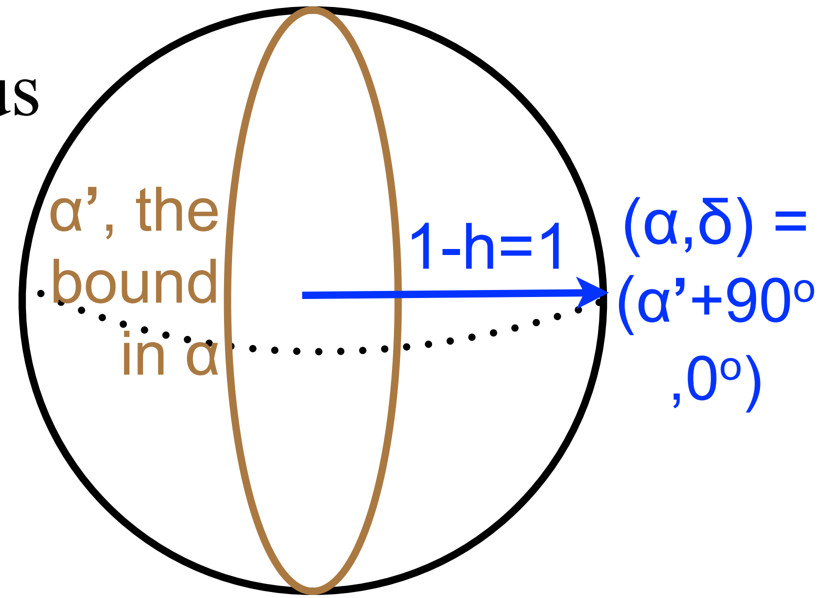
The area of a spherical cap

- Recall that the area of a *lat-lon rectangle* that runs from 0 to 2π in RA is $2\pi(z_2 - z_1)$ where
  - $z_2 - z_1 = \sin\delta_2 - \sin\delta_1$
- For a spherical cap $\delta_2 = 90^o$ and the area is then $2\pi(1 - z_1)$
- But z_1 here is just what I called $|z|$ or h in previous slides
- So, the area of a spherical cap, $2\pi(1-\cos\theta)$ is easy to determine from the vector form for a cap
  - which is $(x,y,z,1-\cos\theta)$



Caps bounded by Right Ascension

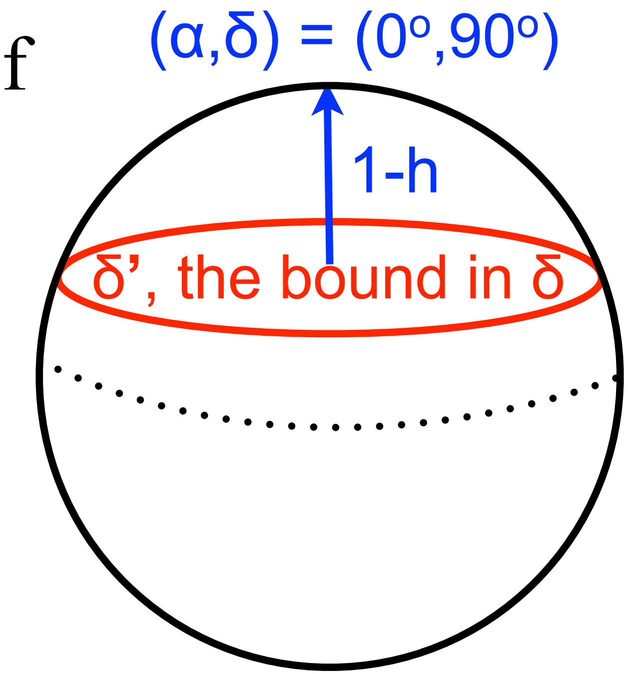
- So far we've discussed the general form, a "circle of radius θ on the surface of the sphere"
- Other main astronomy uses are fields bound by RA or dec
- Bounds in RA (α') map out a great circle on the sphere...a spherical cap that slices off exactly half of the sphere

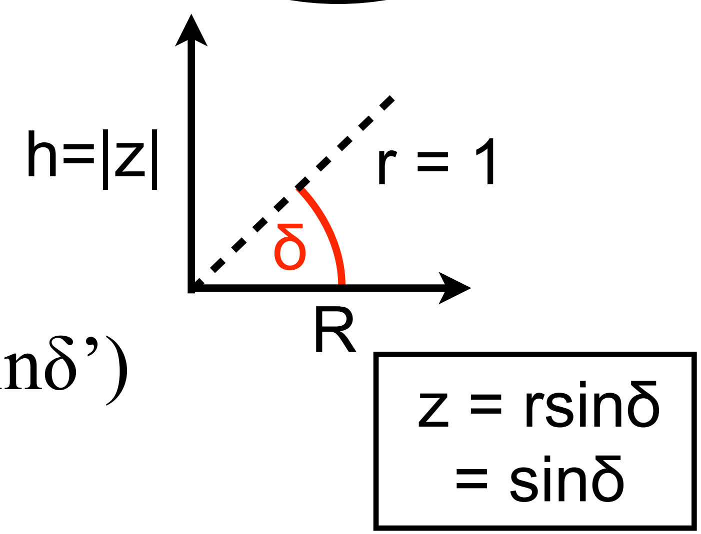


- So, the vector representation of a bound in RA is
  - (x,y,z,1-h) = ($xyz(\alpha'+90^o,0^o)$, 1)
  - By *xyz()* I mean "conversion to Cartesian coordinates"

Caps bounded by Declination

- Bounds in dec (δ') map out lines of constant latitude on the sphere...a cap that slices off increasingly less of the sphere as δ' increases
- The (x,y,z) vector direction is always towards the north pole
- The size of the cap is given by h = sinδ
- So, the vector representation of a bound in declination is
  - $(x,y,z,1\text{-}h) = (\mathit{xyz}(0^o,90^o),\ 1\text{-}\sin\delta')$





Python tasks

1. Write a function to create the vector 4-array for the spherical cap bounded by 5^h in Right Ascension
   - the answer is [-0.96592582629, 0.25881904510, 0, 1]
2. Write a function to create the vector 4-array for the spherical cap bounded by 36^oN in declination
   - the answer is [0, 0, 1, 0.41221474770752686]
3. Write a function to create the vector 4-array for the spherical cap that represents a circular field drawn on the surface of the sphere at $(\alpha,\delta) = (5^h,36^o\mathrm{N})$ with a radius of $\theta = 1^o$
   - the answer is [0.20938900596, 0.78145040877, 0.58778525229,0.00015230484]

Python tasks

4. Write a function that outputs your three spherical caps to a file in the following format:

```
1 polygons
polygon 1 ( 3 caps, 1 weight, 0 pixel, 0 str):
  -0.96592582629 0.25881904510 0 1
  0 0 1 0.41221474770752686
  0.209389006 0.781450409 0.587785253 0.00015230
```

hint:
hdr = 'this text will go first'
np.savetxt(filename, [cap1,cap2], fmt='%1.16f', header=hdr, comments='', newline='\n ')

The reason for the formatting should become clear next week